AKSHAYA INSTITUTE OF TECHNOLOGY

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Module 1 & 2 Notes

Subject: Digital Signal Processing (BEC502)

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Chapter 1

Introduction to Digital Signal Processing

1.1 Introduction

1.1.1 Signal and Signal Processing

A signal is defined as "any physical quantity which varies with one or more independent variables like time, space". Mathematically it can be represented as a function of one or more independent variables. For example, the function

$$s(t) = 2t \qquad ....(1.1.1)$$

describes a signal, which varies linearly with the independent variable t (time). 'Speech' signal is an example, which varies with single independent variable.

Consider the function

$$s(x, y) = 2x + 3y + 5xy \qquad .....(1.1.2)$$

This function describes a signal, which varies with two independent variables x and y. 'Image' is a signal which varies with two independent variables.

Most of the signals encountered are analog in nature i.e., they vary with continuous variable, such as time or space. "Processing of these signals by analog systems such as a filters or frequency analyzers or frequency multipliers for the purpose of changing their characteristics or extracting some desired information is called Signal Processing". In this case both the input signal and the output signal are in analog form, which is shown in Fig. 1.1.

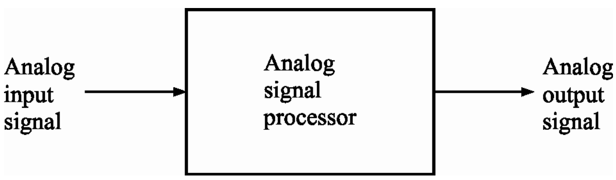


Fig. 1.1 Analog signal processing.

1.1.2 Basic Elements of Digital Signal Processing Systems

Digital Signal Processing is an alternative method to process an analog signal. It requires an interface between an analog input and digital signal processor, called an analog-to-digital (A/D) converter. We should provide another interface between digital signal processor and an analog output signal called a digital-to-analog (D/A) converter as shown in the Fig. 1.2.

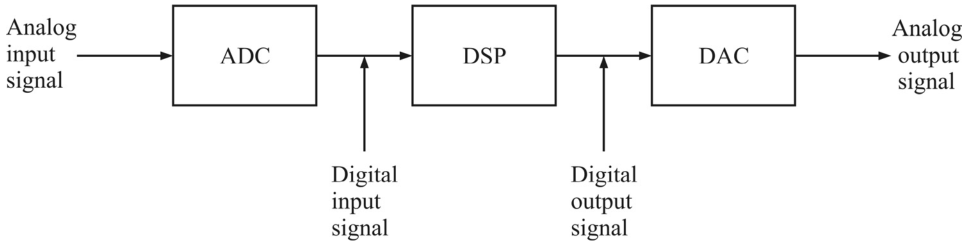


Fig. 1.2 Block diagram of basic DSP system.

The ADC or analog-to-digital (A/D) converter contains a sampler, quantizer and an encoder. Sampler takes an analog signal, samples it with a predefined sampling period and gives out discrete-time signal, which is discrete in time domain and continuous (varying) in amplitude. This signal contains different number of amplitude levels. Quantizer approximates these different levels with fixed number of levels by rounding or truncating the values. For example, if the allowable signal values in the digital signal are integers, say 0 to 7, the continuous-amplitude signal will be quantized into these integer values. Thus the signal value 5.63 will be approximated by the value 6 if the quantization process is performed by rounding to the nearest integer or by 5 if the quantization process is performed by truncation. The encoder converts these set of integer into digital form (i.e., binary form).

The digital signal processor may be a large programmable digital computer or a small microprocessor or a hardwired digital processor to perform the operation on the digital signal i.e., on the output of encoder. Hardwired digital processor performs a specified set of operations on the digital signal i.e., reconfiguring is difficult with the hardwired machines, where as programmable machines provide flexibility to change the operations through a change in the software.

The output of DSP block is digital signal. Digital to analog converter converts digital signal into an analog signal, which may not be required on some applications like extracting information from the radar signal, such as the position of the air-craft and its speed, may simply be printed on paper.

1.1.3 Advantages of Digital Signal Processing over Analog Signal Processing

There are many reasons why digital systems are preferred over an analog system. Some of the advantages are:

(a) *Noise Immunity:* Digital systems are more immune to noise compared to an analog systems.

(b) *Arbitrarily High Accuracy:* Tolerances in analog circuit components make difficult to control the accuracy of an analog systems, where as digital systems provide much better control of accuracy.

(c) *High Reliability:* Digital systems are more reliable compared to an analog systems.

(d) *Software Manipulation:* Digital signal processing operations can be changed by changing the program in digital programmable system, i.e., these are flexible systems.

(e) *Integration of Digital Systems:* Digital systems can be cascaded or integrated easily without any loading problems.

(f) *Storage of Digital Signals:* Digital signals are easily stored on magnetic media such as magnetic tape without loss of quality of reproduction of signal.

(g) *Transportable:* As digital signals can be stored on magnetic tapes these can be processed off time i.e., these are easily transported.

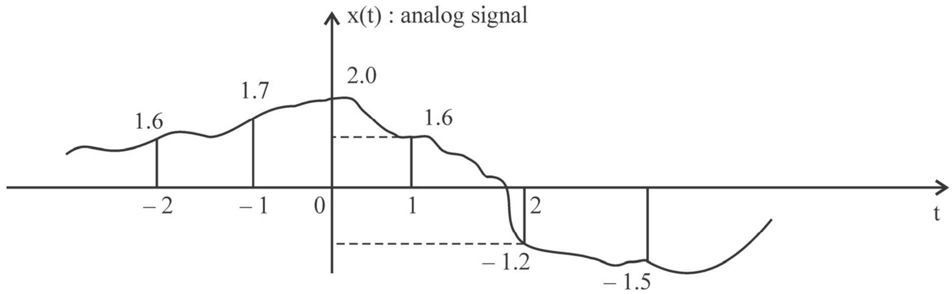
(h) *Digital systems* are independent of temperature, ageing and other external parameters.

(i) *Cheaper:* Cost of processing per signal in DSP is reduced by time-sharing of given processor among a number of signals.

Disadvantage of digital systems is that they are not faster compared to analog systems.

1.2 Discrete Time Signals and Sequences

Discrete-time signals or sequences, which are discrete in time domain and continuous in amplitude, can be obtained by sampling continuous time or analog signals as shown in Fig. 1.3.



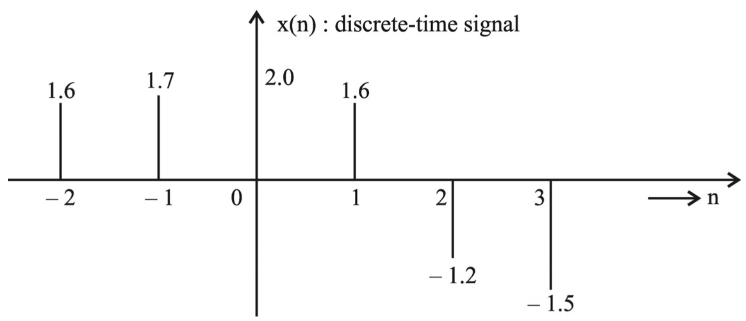


Fig. 1.3 Obtaining discrete-time signal from analog signal.

1.2.1 Representation of Discrete-Time Signals

Discrete-time signals can be represented as follows by using the four methods

(i) Graphical Representation
(ii) Functional Representation
(iii) Tabular Representation
(iv) Sequence Representation

(i) *Graphical Representation:* Discrete-time signals can be represented by a graph when the signal is defined for every integer value of n for $-\infty < n < \infty$. This is shown in Fig. 1.4.

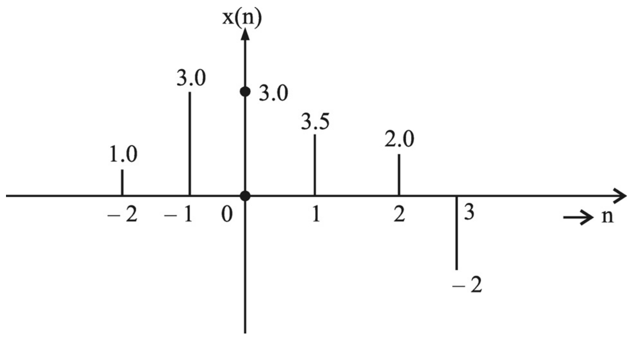


Fig. 1.4 Graphical representation of a discrete-time signal.

(ii) *Functional Representation:* Discrete-time signals can be represented functionally as given below

$$x(n) = \begin{cases} 1, & \text{for } n = 0, 1 \\ 2, & \text{for } n = 2 \\ 3, & \text{for } n = -1 \\ 0, & \text{elsewhere} \end{cases}$$

(iii) *Tabular Representation:* Discrete-time signals can be represented by a table as

n	…..	–2	–1	0	1	2	….
x(n)		1	2	3	1	2	

(iv) *Sequence Representation:* An infinite-duration $(-\infty \le n \le \infty)$ signal with the time as origin (n = 0) and indicated by the symbol ↑, if symbol is not shown in representation, origin is at the beginning of the sequence.

$$x(n) = \left\{ ....1,\ \underset{\uparrow}{2},\ 3, 4, 2, ... \right\}$$

$$x(n) = \{ 2, 3, 1, 4\}$$

Here origin is the first position i.e., x(0) = 2.

1.2.2 Elementary Discrete-Time Signals

There are some basic signals which play an important role in the study of discrete-time signals and systems. These are:

(i) Unit-sample (Cronekar) Sequence, δ(n) (or) Impulse sequence
(ii) Unit-step Sequence, u(n)
(iii) Unit-ramp Sequence, r(n)
(iv) Exponential Sequence

(i) *Unit-sample Sequence:* This is illustrated in Fig. 1.5, it is denoted with δ(n) and is defined as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

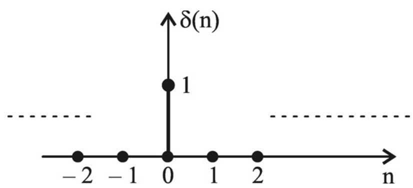


Fig. 1.5 Graphical representation of δ(n).

(ii) *Unit-step Sequence*: It is denoted by u(n) and is defined as

$$u(n) = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

This is shown in Fig. 1.6.

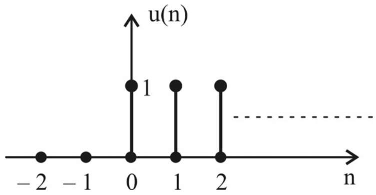


Fig. 1.6 Graphical representation of u(n).

(iii) *Unit-ramp Sequence:* It is denoted by r(n) and is defined as

$$r(n) = \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

This is shown in Fig. 1.7.

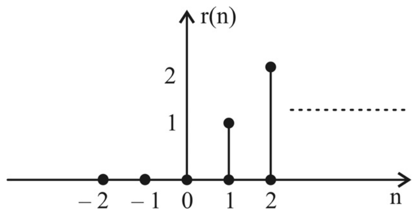
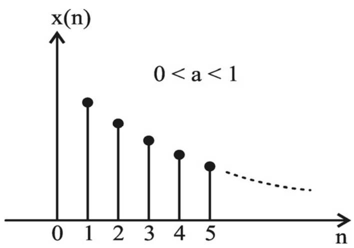


Fig. 1.7 Graphical representation of r(n).

(iv) *Exponential sequence*: It is defined as $x(n) = a^n$ for all values of n.

If the parameter 'a' is real, then x(n) is a real sequence. Fig. 1.8 illustrates this sequence.



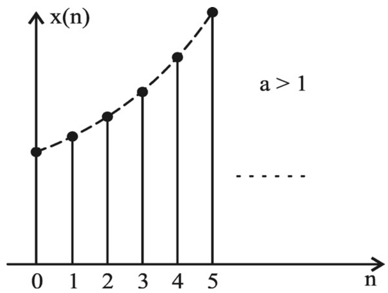


Fig. 1.8 Graphical representation of exponential sequence.

Example 1.1: Let e(n) be an exponential sequence and let x(n) and y(n) denote two arbitrary sequences. Show that

$$[e(n) \,.\, x(n)] * [e(n) \,.\, y(n)] = e(n) \,.\, [x(n) * y(n)]$$

Solution: Given e(n) is an exponential sequence

$\therefore \quad e(n) = a^n$

We know that linear convolution of $x_1(n)$ and x_2 (n) as (* is symbol for linear convolution)

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k).x_2(n-k)$$

In our problem $x_1(n) = e(n) . x(n)$
$= a^n x(n)$

and $x_2(n) = e(n) . y(n)$
$= a^n y(n)$

$$\therefore \quad x_1(n) * x_2(n) = \left[a^n x(n)\right] * \left[a^n\ y(n)\right]$$

$$= \sum_{k=-\infty}^{\infty} a^k x(k).a^{n-k} y(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k).a^k.a^{-k}.a^n y(n-k)$$

$$= a^n \sum_{k=-\infty}^{\infty} x(k) y(n-k)$$

$$\left[a^n\ x(n)\right] * \left[a^n y(n)\right] = a^n . \left[x(n) * y(n)\right]$$

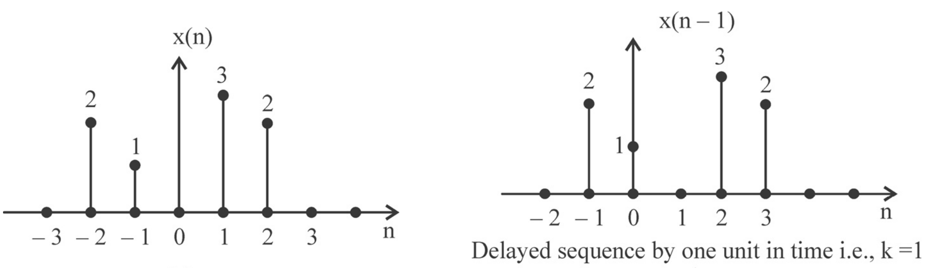
Hence proved

1.2.3 Manipulation of Discrete-Time Signals

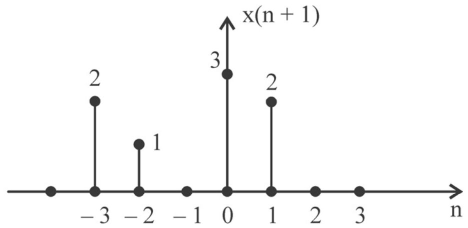
Here we study some simple modifications on independent variable (time) and dependent variable (amplitude of signal). Such modifications are required in DSP techniques. *Modification of the Independent variable (time)*: This can be done in three ways.

(i) Time shifting
(ii) Folding
(iii) Time scaling

(i) *Time Shifting*: A signal can be shifted right side or delayed by replacing n by n – k. and is shifted left side or advanced by replacing n by n + k, where k is integer and n is a discrete-time index. This is shown in Fig. 1.9.



(a) (b)

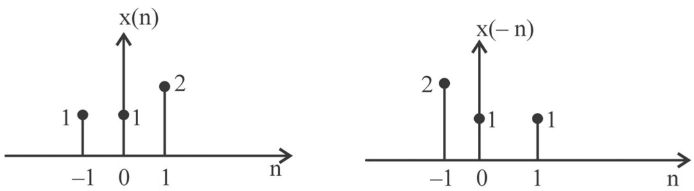


Advance sequence by one unit in time i.e., k =1

(c)

Fig.1.9 (a) Original sequence (b) Delayed by one unit version of original sequence (c) Advanced sequence by one unit version of original sequence.

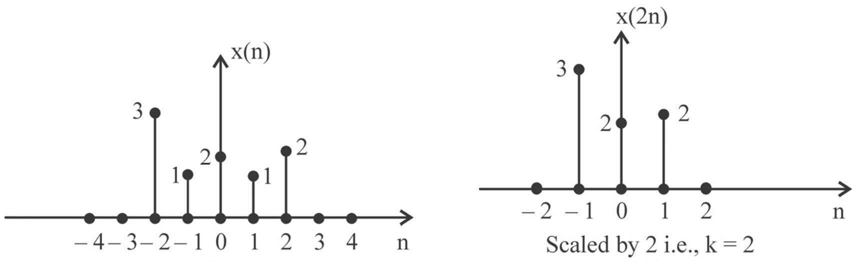
(ii) *Folding:* If independent variable (time) n is replaced by – n, then signal folding (mirror image) about the time origin (n = 0) takes place. This is shown in Fig. 1.10.



(a) (b)

Fig. 1.10 (a) Original sequence (b) Folded version of original sequence.

(iii) *Time Scaling:* Time scaling is performed by replacing independent variable n by kn, where k is an integer. This is shown in Fig. 1.11.



(a) Scaled by 2 i.e., k = 2 (b)

Fig. 1.11 (a) Original sequence (b) Scaled version of original sequence.

***Modification of the Dependent Variable (Signal Amplitude)*:**

Signal amplitude can be modified in three ways.

(i) Addition of sequences

(ii) Multiplication of sequences

(iii) Amplitude scaling of sequence

(i) *Addition of Sequences:* The sum of two discrete time sequences is given by

$$y(n) = x_1(n) + x_2(n), \qquad \infty < n < \infty$$

This is shown in Fig. 1.12 (a)

(ii) *Multiplication of Sequences:* The product of two discrete time sequences is given by

$$y(n) = x_1(n) \,.\, x_2(n), \qquad -\infty < n < \infty$$

This is shown in Fig. 1.12 (b)

(iii) *Amplitude Scaling of Sequences:* Amplitude scaling of a signal by a constant A is accomplished by multiplying the value of every signal sample by A.

$$y(n) = A\, x(n), \qquad -\infty < n < \infty$$

where A is real constant quantity

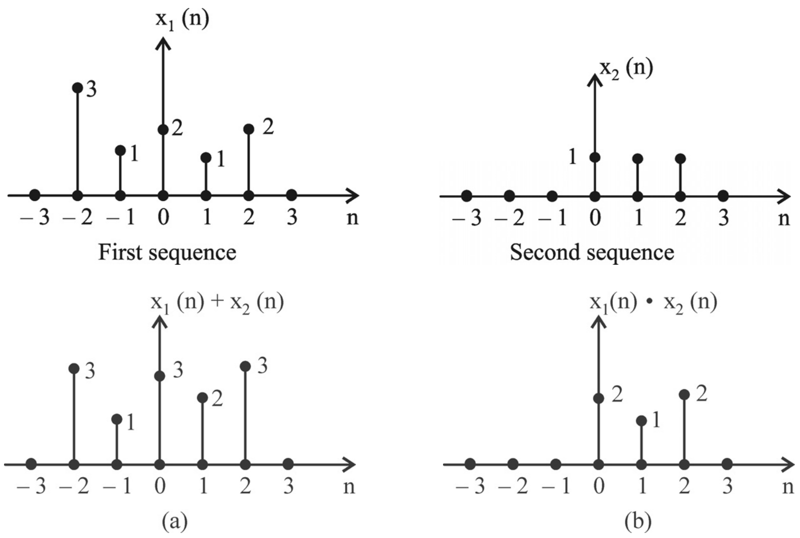


Fig. 1.12 (a) Addition of sequences (b) Multiplication of sequences.

1.2.4 Classification of Discrete-Time Signals

Discrete-time signals are classified based on number of different characteristics as follows:

(i) Energy signals and power signals
(ii) Periodic signals and Aperiodic signals
(iii) Symmetric (Even) and Antisymmetric (odd) signals.

(i) *Energy Signals and Power Signals*

The energy E of a signal x(n) is defined as $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$ …..(1.2.1)

Here x(n) may be either complex or real valued signal; E may be finite or infinite.

If E is finite, then x(n) is called energy signal. Many signals that posses infinite energy have a finite average power.

Average power of a signal x(n) is defined as

$$P_{av} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \quad \text{.....(1.2.2)}$$

If E is finite, $P_{av} = 0$

If E is infinite, P_{av} may be either finite or infinite if P_{av} is finite, then x(n) is called a power signal.

(ii) *Periodic Signals and Aperiodic Signals*

If a signal x(n) satisfies the condition x(n) = x(n + N), where N is period then the signals is periodic signal, otherwise it is nonperiodic or aperiodic signal.

Consider a sinusoidal signal $\cos(\omega_0 n + \phi)$, it will be periodic only if $\frac{2\pi}{\omega_0}$ is an integral number. If $\frac{2\pi}{\omega_0}$ is a rational, then the function will have a period longer than $\frac{2\pi}{\omega_0}$.

If $\frac{2\pi}{\omega_0}$ is not a rational number, it will not be periodic at all.

The energy of a periodic signal over a single period, say $0 \le n \le N - 1$, is finite, but energy of periodic signal for $-\infty \le n \le \infty$ is infinite. On the other hand, the average power of the periodic signal is finite.

∴ Periodic signals are power signals.

(iii) *Even signals and Odd signals*

A real valued signal x(n) is called symmetric (even)

If $x(n) = x(-n)$ …..(1.2.3)

On the other hand, a signal x(n) is called anti symmetric (odd),

if $x(-n) = -x(n)$ …..(1.2.4)

Any real sequence can be written as

$$x(n) = x_o(n) + x_e(n)$$

where $x_o(n)$ is odd part of x(n)

and $x_e(n)$ is even part of x(n)

$x_e(n)$ can be written as

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right] \qquad \text{.....(1.2.5)}$$

and $$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

Similarly for complex sequence

Condition for symmetry is x(n) = x*(– n)

where * denotes conjugation

And for Anti-symmetry is $x(n) = -x^*(-n)$

Any complex sequence can be written as

$$x(n) = x_o(n) + x_e(n)$$

where $$x_e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right] \qquad \text{......(1.2.6)}$$

and $$x_o(n) = \frac{1}{2}\left[x(n) - x^*(-n)\right]$$

Example 1.2: Show that the even and odd parts of a real sequence are, respectively, even and odd sequences. [JNTU 2002]

Solution*:* Let x(n) be real sequence, which can be written as

$$x(n) = x_o(n) + x_e(n)$$

where $x_o(n)$ is odd part of x(n) and $x_e(n)$ is even part of x(n)

We know that $x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$ and $x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$

We have to show that even part ($x_e(n)$) of x(n) is a even sequence i.e., it should satisfy $x_e(n) = x_e(-n)$

Consider $$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

take $$x_e(-n) = \frac{1}{2}[x(-n) + x(n)]$$

$$= \frac{1}{2}[x(n) + x(-n)]$$

$$x_e(-n) = x_e(n)$$

Hence proved.

Similarly we have to show that odd part $(x_o(n))$ of x(n) is a odd sequence i.e., it should satisfy $x_o(n) = -x_o(-n)$

Consider $$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

take $$x_o(-n) = \frac{1}{2}[x(-n) - x(n)]$$

$$= -\frac{1}{2}[x(n) - x(-n)]$$

$$x_o(-n) = -x_o(n)$$

Hence proved.

1.3 Linear Shift Invariant System, Stability and Causality

Before going to discuss linear shift invariant systems, stability and causality, let us define discrete time system. A discrete time system is a device or an algorithm that operates on a discrete time signal, called the input or excitation, according to some well-defined rule, to produce another discrete time signal called the output or response of the system.

We say that the input signal x(n) is transformed by the system into a signal y(n). These two can be related as

$$y(n) \equiv H[x(n)] \quad \text{.....(1.3.1)}$$

This is shown graphically in Fig. 1.13.

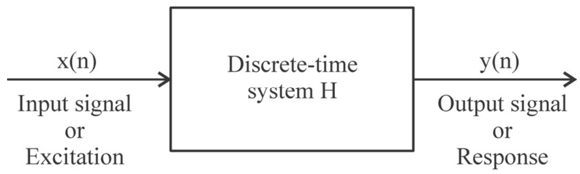


Fig. 1.13 Discrete time system.

1.3.1 Basic Building Blocks of a Discrete Time System

There are three basic building blocks of a discrete time system.

(i) Adders or summing element

(ii) Multipliers

(iii) Delay Elements

(i) *Adders:* It performs addition of two or more discrete-time signals as shown in Fig. 1.14(a).

(ii) *Multipliers*: There are two types of multipliers (a) constant multiplier (b) signal multiplier. A signal multiplier performs multiplication of two or more discrete-time signals as shown in Fig. 1.14(b).

A constant multiplier performs multiplication of a discrete-time signal with a scalar quantity as shown in Fig 1.14(c).

(iii) *Delay Elements:* There are two types of delay elements

(a) positive delay element, which is indicated by Z^{-1} (b) negative delay elements, which is indicated by Z^{+1} (or) advance element. Positive and negative delay elements provide delay as shown in Fig. 1.14(d) and (e) respectively.

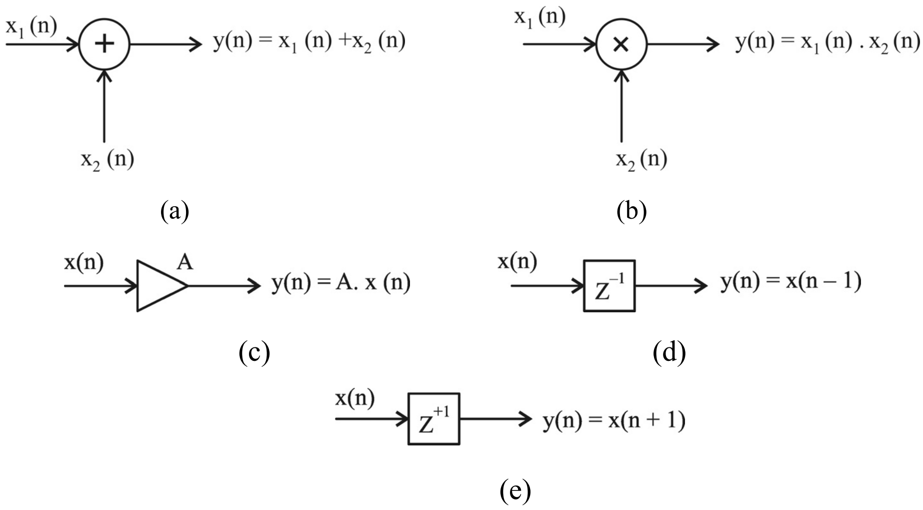


Fig. 1.14 (a) Adder (b) A signal multiplier (c) A constant multiplier (d) A unit delay element (e) A unit advance element.

1.3.2 Classification of Discrete-Time Systems

Discrete-time systems can be classified into

(i) Static (memory less) systems and Dynamic (systems with memory)systems.

(ii) Time-invariant and Time-varying systems
(iii) Linear systems and Non-linear systems
(iv) Causal systems and Non-causal systems
(v) Stable systems and Unstable systems.

(i) *Static and Dynamic Systems*: Static systems are also called memory less systems. A discrete-time system is called memory less system if its output at any instant n depends at most on the input at the same instant, but not on past or future values of input samples.

Example:

$$\left.\begin{aligned} y(n) &= A.x(n) \\ y(n) &= n\ x(n) + Ax^2(n) \end{aligned}\right\} \text{Both systems are static systems}$$

On the other hand, output of a system which depends on past or future samples of the input signal is called dynamic system. It is also called a system with memory. These systems require memory for storage for future and past samples of input signal.

Example:

$$y(n) = x(n) + x(n+1) + x(n-1)$$

is a dynamic system.

(ii) *Time-invariant and Time-varying systems*: A system is called time-invariant if its input-output characteristics do not change with time.

If the response to a delayed input, and the delayed response are equal then the system is called time-invariant system (or) shift invariant system.

The response to a delayed input is denoted by $y(n,\ k)$ and the delayed response is denoted by $y(n-k)$. If both responses $y(n,k)$ and $y(n-k)$ are equal then the system is called time-invariant system. If both responses are not equal then the system is called time-varying system.

Example 1.3: Check the following system for Time-invariance

$$y(n) = x(n) + n\ x(n-1)$$

Solution*:* The response to a delayed input is

$$y(n,\ k) = x(n-k) + n\ x(n-k-1)$$

The delayed response is

$$y(n-k) = x(n-k) + (n-k)x(n-k-1)$$

both responses are not equal

i.e., $y(n,k) \neq y(n-k)$

Therefore the given system is not a Time-invariant system. It is a Time-varying system.

Example 1.4: Check the following system for Time-invariance

$$y(n) = x(n) + x(n-1)$$

Solution*:* The response to a delayed input is

$$y(n, k) = x(n-k) + x(n-1-k)$$

The delayed response is

$$y(n-k) = x(n-k) + x(n-k-1)$$

both responses are equal. Hence given system is a Time-invariant system.

(iii) *Linear and Non-linear Systems*: A system which satisfies superposition principle is called a linear system. A system which does not satisfy superposition principle is termed as a non-linear system.

Superposition principle is stated as

"Response of the system to a weighted sum of input signals be equal to the corresponding weighted sum of responses of the system to each of the individual input signals".

A system is linear if and only if

$$H[ax_1(n) + bx_2(n)] = aH[x_1(n)] + b\ H[x_2(n)] \qquad \text{.....(1.3.2)}$$

where $x_1(n)$ and $x_2(n)$ are arbitrary input signals and a and b are arbitrary constants.

Example 1.5: Check the following systems for Linearity.

(i) $y(n) = x(n^2)$ (ii) $y(n) = e^{x(n)}$

Solution*:*

(i) The corresponding outputs for two discrete-time sequences $x_1(n)$ and $x_2(n)$ are

$$y_1(n) = x_1(n^2)$$

$$y_2(n) = x_2(n^2)$$

A linear combination of two input sequences results in the output

$$y_3(n) = H[x_3(n)] = H[a\ x_1(n) + b\ x_2(n)]$$

$$= x_3(n^2) = a\ x_1(n^2) + b\ x_2(n^2) \qquad \text{.....(i)}$$

A linear combination of the two outputs results in the output

$$a\ y_1(n) + b\ y_2(n) = a\ x_1(n^2) + b\ x_2(n^2) \qquad \text{.....(ii)}$$

Since both outputs are equal, the system is linear.

(ii) The corresponding outputs for two discrete-time sequences $x_1(n)$ and $x_2(n)$ are

$$y_1(n) = e^{x_1(n)}$$

$$y_2(n) = e^{x_2(n)}$$

A linear combination of $x_1(n)$ and $x_2(n)$ results in the output.

$$y_3(n) = H\,[x_3(n)] = H[a\ x_1(n) + b\ x_2(n)] = e^{ax_1(n)+bx_2(n)} \qquad \text{.....(iii)}$$

Linear combination of the two outputs results in the output

$$a\ y_1(n) + b\ y_2(n) = ae^{x_1(n)} + b\ e^{x_2(n)} \qquad \text{.....(iv)}$$

Here both outputs i.e., equations (iii) and (iv) are not equal, hence the system is non-linear.

(iv) *Causal and Non-causal Systems*: A system whose present output depends only on present and past inputs, but not on future inputs is called a causal system. If a system response depends on future values, then it is a non-causal system.

Example:

(i) $y(n) = x(n) + x(n+1)$

Since response depends upon a future value $(x(n+1))$, it is a non-causal system.

(ii) $y(n) = x(n) + x(n-1)$

Since response does not depend upon future values, it is a causal system.

(iii) $y(n) = x(-n)$

Take n = –1, then y(–1) = x(1), which is a future value i.e., it depends on future values. Hence the system is a non causal system.

(iv) $y(n) = x(n^2)$ and $y(n) = x(2n)$

Take n = 2, then y(2) = x(4) in both the systems which is a future value. Hence both systems are non causal systems.

(v) *Stable and Unstable Systems:* A system which produces bounded (finite) output for a bounded (finite) input is called as stable system, otherwise it is called as an unstable system. Examples will be discussed in the section 1.3.5.

1.3.3 Representation of Discrete Time Signal as Summation of Impulses

Graphical representations of impulse sequences and its shifted versions are shown in Fig. 1.15 (a) (b) and (c).

Let us consider a sequence

$$x(n) = \left\{1, \underset{\uparrow}{1}, 2, 3\right\}$$

which is shown graphically in fig 1.15 (d)

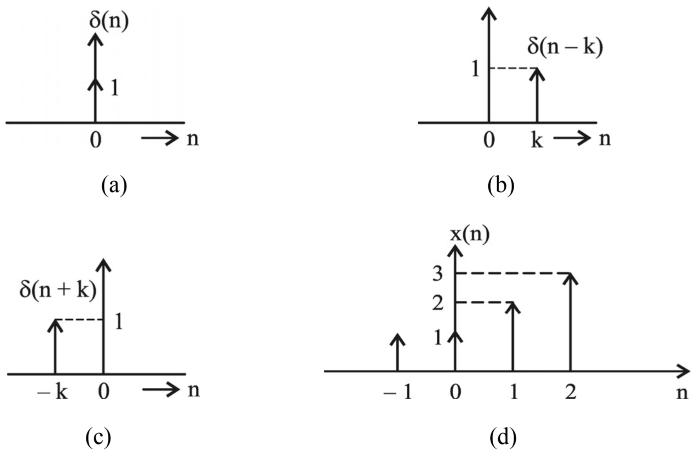


Fig. 1.15 (a) Impulse sequence (b) Shifted towards right (c) Shifted towards left (d) A general sequence.

product of x(n) and δ(n) gives x(0)

i.e., $x(n)\,\delta(n) = x(0) \rightarrow x(0)\,\delta(n) = x(0)$

similarly $x(n)\,\delta(n-1) = x(1) \rightarrow x(1)\,\delta(n-1) = x(1)$

$x(n)\,\delta(n-2) = x(2) \rightarrow x(2)\,\delta(n-2) = x(2)$

$x(n)\,\delta(n+1) = x(-1) \rightarrow x(-1)\,\delta(n+1) = x(-1)$

∴ This can be written as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad \text{…..(1.3.3)}$$

Example: Represent the sequence $x(n) = \left\{2, \underset{\uparrow}{3}, 5\right\}$ as sum of impulse sequences

Solution is $x(n) = 2\,\delta(n+1) + 3\,\delta(n) + 5\,\delta(n-1)$

1.3.4 Response of Linear Time Invariant (LTI) System

Fig. 1.16 shows an LTI system with an excitation x(n) and response y(n).

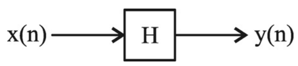


Fig. 1.16 A discrete time LTI system.

Unit sample response (or) Impulse response: It is defined as "response of a system when input signal is impulse sequence", impulse response is denoted with h (n)

i.e., when $x(n) = \delta(n) \rightarrow y(n) = h(n)$

Since it is a time invariant system, when impulse sequence is delayed by k, response i.e., h (n) should be delayed by k.

$\therefore$ $\delta(n-k) \rightarrow h(n-k)$

This also can be written as $h(n-k) = H[\delta(n-k)]$

Response of an LTI (Linear Time Invariant) system is

$$y(n) = H[x(n)] \qquad \text{.....(1.3.4)}$$

we known that x(n) can be represented as sum of impulse sequence as in eqn. (1.3.3).

i.e., $$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Substitute this in the equation (1)

$$\therefore \quad y(n) = H\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, H[\delta(n-k)]$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad \text{.....(1.3.5)}$$

which is an equation for convolution of x (n) and h (n).

Hence the response of an LTI system is convolution of input sequence and impulse response.

Example 1.6: Consider a discrete linear shift invariant system with unit sample response h(n). If the input x(n) is a periodic sequence with period N i.e., $x(n) = x(n + N)$, show that the output y(n) is also a periodic sequence with period N.

Solution: Given $x(n) = x(n + N)$

We known the response of an LTI (or) Liner shift invariant system as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

also $$y(n) = \sum_{k=-\infty}^{\infty} h(x)x(n-k)$$

consider $$y(n+N) = \sum_{k=-\infty}^{\infty} h(k)x(n+N-k)$$

given $x(n) = x(n+N)$

delay it by k

then $x(n-k) = x(n+N-k)$

$$\therefore \quad y(n+N) = \sum_{k=-\infty}^{\infty} h(x)x(n-k)$$

$$= y(n)$$

Hence the output is also periodic when input is periodic

1.3.5 Stability of an LTI System

Let x(n) is input sequence, assume that it is finite with a value M_x.

Response of an LTI system is

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Take absolute on both sides

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)|\ |x(n-k)|$$

$$\le M_x \sum_{k=-\infty}^{\infty} |h(k)| \qquad \text{.....(1.3.6)}$$

According to definition for stability, for a finite input sequence, system should produce finite output.

From eqn (1.3.6), to get finite output, $\sum_{k=-\infty}^{\infty} |h(k)|$ must be finite

$$\therefore \quad \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Hence an LTI system is stable if its impulse response (or) unit sample response is absolutely summable.

Example 1.7: Test the stability of the following systems

(i) $y(n) = x(-n-2)$ (ii) $y(n) = n\,x(n)$

Solution: We know that when $x(n) = \delta(n)$, the output $y(n) = h(n)$

(i) $\therefore$ $h(n) = \delta(-n-2)$

$$n = 0 \rightarrow \delta(-2) = 0 = h(0);\; n = -1 \rightarrow \delta(-1) = 0 = h(-1)$$

$$n = 1 \rightarrow \delta(-3) = 0 = h(1);\; n = -2 \rightarrow \delta(0) = 1 = h(-2)$$

$$\vdots \qquad\qquad \vdots$$

$$n = \infty \rightarrow \delta(\infty) = 0 = h(\infty);\; n = -\infty \rightarrow \delta(-\infty) = 0 = h(-\infty)$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| = 0 + 0 + \cdots\cdots + 1 + 0 + \cdots\cdots + 0 = 1$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$\therefore$ System is stable

(ii) $h(n) = n\,\delta(n)$

$$n = 0 \rightarrow h(0) = 0(1) = 0;\;\; n = -1 \;\rightarrow\; h(-1) = -1(0) = 0$$

$$n = 1 \rightarrow h(1) = 1(0) = 0;\;\; n = -2 \;\rightarrow\; h(-2) = -2(0) = 0$$

$$\vdots \qquad\qquad \vdots$$

$$n = \infty \rightarrow h(\infty) = \infty(0) = 0;\;\; n = -\infty \;\rightarrow\; h(-\infty) = -\infty(0) = 0$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| = 0 + 0 + \cdots\cdots + 0 = 0$$

System is stable

Example 1.8: Determine the range of values of the parameter 'a' for which the LTI system with impulse response $h(n) = a^n u(n)$ is stable.

Solution: Condition for a system to be stable is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$\therefore$ $$\sum_{n=0}^{\infty} |a^n| = 1 + |a| + |a|^2 + \cdots\cdots$$

This infinite series converges to

$$= \frac{1}{1-|a|} \text{ if } |a| < 1$$

$\therefore$ Range of values of parameter 'a' is $|a| < 1$

Example 1.9: Determine the range of values of 'a' and 'b' for which LTI system with impulse response $h(n) = \begin{cases} a^n, & n \geq 0 \\ b^n, & n < 0 \end{cases}$ is stable

Solution: $$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{-1} b^n + \sum_{n=0}^{\infty} a^n$$

Put n = – 1 in first series

$$= \sum_{n=1}^{\infty} \frac{1}{b^n} + \sum_{n=0}^{\infty} a^n$$

$$= \left(\frac{1}{b} + \frac{1}{b^2} + \cdots\cdots \right) + \left(1 + a + a^2 + \cdots\cdots \right)$$

$$= \frac{1}{b}\left(1 + \frac{1}{b} + \frac{1}{b^2} + \cdots\cdots \right) + \left(1 + a + a^2 + \cdots\cdots \right)$$

$$= \frac{1}{b}\left(\frac{1}{1 - \frac{1}{b}} \right) + \frac{1}{1-a} \quad \text{if } \begin{cases} \frac{1}{b} < 1 \text{ and } a < 1 \\ b > 1 \text{ and } a < 1 \end{cases}$$

$\therefore$ Range of values of 'a' and 'b' are $b > 1$ and $a < 1$

Example 1.10: A unit sample response of a linear system is given by $h(n) = (n+b)a^n, \ n \geq 0$

$= 0, \ n < 0$

For what values of 'a' and 'b' the system will be stable?

Solution: $$\sum_{n=0}^{\infty} (n+b)a^n = \sum_{n=0}^{\infty} n \ a^n + \sum_{n=0}^{\infty} b \ a^n$$

$$= \left(0 + a + 2a^2 + 3a^3 + \cdots\cdots \right) + b\left(1 + a + a^2 + \cdots\cdots \right)$$

$$= a\left(1 + 2a + 3a^2 + \cdots\cdots \right) + b\frac{1}{1-a}$$

$$= a\frac{1}{(1-a)^2} + \frac{b}{1-a}$$ if $a < 1$ and b must finite to become series finite

∴ Values of 'a' and 'b' are $a < 1$ and $b < \infty$

1.4 Linear-Constant Coefficient Difference Equations

We know that continuous time systems are described by differential equations. But discrete-time systems are described by difference equations.

Input-output relation of N^{th} order discrete-time system can be written as

$$\sum_{k=0}^{N} a_k \; y(n-k) = \sum_{k=0}^{M} b_x \; x(n-k) \qquad \text{.....(1.4.1)}$$

where y(n) is output

x(n) is input

and a_k and b_k are constant coefficients. Order of the system is determined by L.H.S summation since input-output relation is linear with constant coefficients, this equation is called "Linear-constant coefficient difference equation" for N^{th} order.

There are two methods by which difference equations can be solved

1. **Direct Method:** This method is directly applicable in the time domain. We are not discussing this method
2. **Indirect Method:** It is also called z-transform method. This method will be discussed in the chapter 3.

1.5 Frequency Domain Representation of Discrete-Time Systems and Signals

1.5.1 Frequency Domain Representation of Discrete-Time System

System function (or) transfer function of a system can be obtained by taking Z-transform (for Z-transform refer chapter 3) of impulse response h(n)

i.e., system function = $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$ ....(1.5.1)

system function can also be defined as ratio of z-transform of response to z-transform of input with zero initial conditions.

i.e., $$H(z) = \frac{Y(z)}{X(z)} \qquad \text{....(1.5.2)}$$

Frequency response of a system can be obtained just by putting $z = e^{j\omega}$ in equation (1.5.1).

i.e., $$H(e^{j\omega}) = H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} \qquad \text{....(1.5.3)}$$

Magnitude spectrum of a system is obtained by taking modulus of $H(e^{j\omega})$ i.e., $|H(e^{j\omega})|$ Phase spectrum of a system is obtained by

$$\theta = \tan^{-1}\left(\frac{H_i(\omega)}{H_r(\omega)}\right) \qquad \text{....(1.5.4)}$$

where $H_i(\omega)$ = Imaginary part of $H(\omega)$

$H_r(\omega)$ = Real part of $H(\omega)$

1.5.2 Frequency Domain Representation of Discrete-Time Signals

Let us consider any discrete-time sequence say x(n)

Frequency domain representation of this sequence can be obtained by taking z-transform of x(n) and putting $z = e^{j\omega}$.

i.e., $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where x(z) is z-transform of x(n)

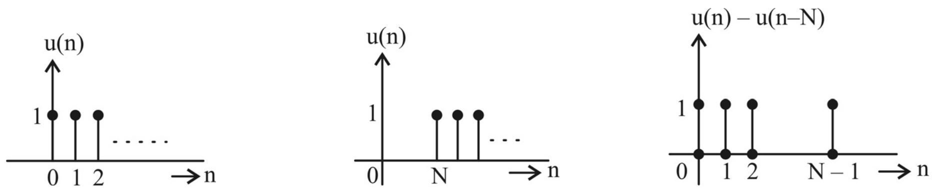
put $z = e^{j\omega}$

$$\therefore \quad X(e^{j\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

which is frequency domain representation of x(n).

Example 1.11: An LTI system has unit sample response h(n) = u(n) – u(n–N). Find the amplitude and phase spectra.

Solution:



$\therefore \quad h(n) = 1, \quad n = 0$ to $N - 1$

from the figures shown

$$H(z) = \sum_{n=0}^{N-1} 1.\, z^{-n}$$

$$= \sum_{n=0}^{N-1} z^{-n}$$

$$= \frac{1 - z^{-N}}{1 - z^{-1}}$$

$$\because \sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a}$$

Finite Geometric Series

Frequency respo... $z = e^{j\omega}$

$$H(\omega) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{e^{\frac{-j\omega N}{2}}\left(e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}}\right)}{e^{\frac{-j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)}$$

$$= e^{\frac{-j\omega}{2}(N-1)} \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Magnitude spectra is

$$|H(\omega)| = \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Phase spectra is

$$\angle H(\omega) = \theta = -\frac{\omega}{2}(N - 1)$$

Review Questions

1. Pick the signal which varies with single independent variable.

   (a) Speech (b) Image (c) $S = 2x + 6xy$ (d) $S = 3x^2y$ *Ans*: [b]

2. Pick the signal which varies with two independent variables.

   (a) Image (b) $S = 2t^2$ (c) Speech (d) $S = 2t + 3t^3$

   Ans: [c]

3. Which of the following is not an analog system?
   (a) Frequency analyzers (b) Analog filters
   (c) Frequency multipliers (d) Programmable machines
   Ans: [d]
4. Which of the following is not a block in basic DSP system?
   (a) ADC (b) Digital signal processor
   (c) Analog signal processor (d) DAC *Ans*: [c]
5. Which of the following is not a part of an analog-to-digital converter?
   (a) Sampler (b) Decoder (c) Encoder (d) Quantizer *Ans*: [b]
6. The following is the disadvantage of digital systems
   (a) Cost (b) Speed (c) Transportability (d) Noise immunity
   Ans: [b]
7. Discrete-time signal is
   (a) Discrete both in time and amplitude
   (b) Discrete in time and continuous in amplitude
   (c) Continuous in time and discrete in amplitude
   (d) All the above *Ans*: [b]
8. Digital signal is
   (a) Discrete both in time and amplitude
   (b) Continuous in time and discrete in amplitude
   (c) Discrete in time and continuous in amplitude
   (d) All the above *Ans*: [a]
9. Discrete-time signal can be represented by
   (a) Graphical method (b) Functional method
   (c) Sequence method (d) All the above *Ans*: [d]
10. The other name of unit impulse sequence
    (a) Unit-sample sequence (b) Unit-step sequence
    (c) Unit ramp sequence (d) All the above *Ans*: [a]
11. Unit step sequence is defined as
    (a) $x(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ (b) $x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
    (c) $x(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$ (d) $x(n) = \begin{cases} 0 & n \geq 0 \\ 1 & n < 0 \end{cases}$ *Ans*: [b]

12. Unit impulse is defined as

(a) $x(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ (b) $x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

(c) $x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ (d) $x(n) = \begin{cases} 0 & n \geq 0 \\ 1 & n < 0 \end{cases}$ *Ans*: [a]

13. Unit ramp is defined as

(a) $x(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ (b) $x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

(c) $x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ (d) $x(n) = \begin{cases} 0 & n \geq 0 \\ 1 & n < 0 \end{cases}$ *Ans*: [c]

14. Exponential sequence a^n is decaying when

(a) $0 < a < 1$ (b) $a > 1$ (c) $a < 1$ (d) $-1 < a < 1$ *Ans*: [a]

15. Discrete-time signal can be modified by modifying independent variable using the following methods.

(a) Time shifting (b) Folding

(c) Time-scaling (d) All the above *Ans*: [d]

16. A signal x(n) can be shifted right side by replacing n with

(a) $n + k$ (b) $n - k$ (c) $n \div k$ (d) nk *Ans*: [b]

17. A signal x(n) can be shifted left side by replacing n with

(a) $n + k$ (b) $n - k$ (c) $n \div k$ (d) nk *Ans*: [a]

18. A signal x(n) will be folded if n is replaced by

(a) $-n$ (b) $\div n$ (c) $n + k$ (d) $n - k$ *Ans*: [a]

19. Discrete-time signal can be modified by modifying dependant variable using the following methods

(a) Addition of sequences (b) Multiplication of sequences

(c) Amplitude scaling of sequence (d) All the above *Ans*: [d]

20. Energy of a signal is

(a) $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$ (b) $E = \frac{1}{N}\sum_{n=-\infty}^{\infty} |x(n)|^2$

(c) $E = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x(n)|^2$ (d) $E = \frac{1}{2N}\sum_{n=-N}^{N} |x(n)|^2$ *Ans:* [a]

21. Average power of a signal is

(a) $P_{av} \quad \underset{N}{Lim} \frac{1}{2N \quad 1} \; {}_{n \quad N}^{N} |x(n)|^2$ (b) $P_{av} \quad \frac{1}{N} \; {}_{n} \quad |x(n)|^2$

(c) $P_{av} \quad {}_{n} \quad |x(n)|^2$ (d) $P_{av} \quad \frac{1}{N} \; {}_{n \quad N}^{N} |x(n)|^2$ *Ans*: [a]

22. Energy singal's average power is

(a) Infinite (b) Zero

(c) Cannot be determined (d) None *Ans*: [b]

23. Power signal's is energy is

(a) Infinite (b) Zero

(c) Cannot be determined (d) None *Ans*: [a]

24. A periodic signal satisfies the condition

(a) $x(n) \quad x(n \quad N)$ (b) $x(n) \quad x(n \quad N)$

(c) $x(n) \quad x(\quad n)$ (d) $x(n) = -x(-n)$ *Ans*: [a]

25. A sinusoidal signal $\cos \quad {}_0 n$ will be periodic only if $\frac{2}{\quad_0}$ is an/a

(a) Integer (b) Irrational (c) Infinite (d) Zero

Ans: [a]

26. A real signal x(n) is called symmetric if

(a) $x(n) \quad x(n \quad N)$ (b) $x(n) \quad x(\quad n)$

(c) $x(n) \quad\quad x(\quad n)$ (d) $x(n) \neq x(n+N)$ *Ans*: [b]

27. A real signal x(n) is called anti symmetric if

(a) $x(n) \quad x(n \quad N)$ (b) $x(n) \quad x(\quad n)$

(c) $x(n) \quad\quad x(\quad n)$ (d) $x(n) \neq x(n+N)$ *Ans*: [c]

28. A complex signal x(n) is called symmetric if

(a) $x(n) \quad x(n \quad N)$ (b) $x(n) \quad x*(\quad n)$

(c) $x(n) \quad x(\quad n)$ (d) $x(n) = -x(-n)$ *Ans*: [b]

29. A complex signal x(n) is called odd signal if

(a) $x(n) = -x*(-n)$ (b) $x(n) = -x(-n)$

(c) $x(n) = x*(-n)$ (d) $x(n) = x(-n)$ *Ans*: [a]

30. The other name of LTI system is

(a) LTV (b) LSI (c) TV (d) None

Ans: [b]

31. Positive delay element is

(a) Z^{-1} (b) Z^{+1} (c) Z (d) None

Ans: [a]

32. Negative delay element is

(a) Z^{+1} (b) Z^{-1} (c) $\frac{1}{Z}$ (d) None

Ans: [a]

33. The other name of static system

(a) Causal (b) Memory less

(c) Time variant (d) Stable *Ans*: [b]

34. The other name of Dynamic system

(a) Causal (b) Memory less

(c) Time variant (d) System with memory *Ans*: [d]

35. Linear system should satisfy

(a) Stability condition (b) $y(n-k) \leftrightarrow x(n-k)$

(c) Superposition principle (d) None *Ans*: [c]

36. Time invariant system should satisfy

(a) Stability condition (b) $y(n-k) \leftrightarrow x(n-k)$

(c) Superposition principle (d) None *Ans*: [b]

37. Causal system response depends upon

(a) Future input

(b) Present input, past input, future input

(c) Past input, future input

(d) Present input, past input *Ans*: [d]

38. Pick a causal system

(a) $y(n) = x(-n)$ (b) $y(n) = x(2n)$

(c) $y(n) = x(n)$ (d) $y(n) = x(n^2)$ *Ans*: [c]

39. Pick non causal system
    (a) $y(n) = x(-n)$ (b) $y(n) = x(n) + x(n-1)$
    (c) $y(n) = 2x(n)$ (d) $y(n) = x(n-1)$ *Ans*: [a]

40. Pick non causal system
    (a) $y(n) = x(n)$ (b) $y(n) = x(n) + x(n+1)$
    (c) $y(n) = x(n) + x(n-1)$ (d) $y(n) = x(n-2)$ *Ans*: [b]

41. A system is said to be unstable if it gives ….. output, for a finite input
    (a) Finite (b) Zero (c) Infinite (d) One
    Ans: [c]

42. Condition for a system to be stable is
    (a) Impulse response should be absolutely summable
    (b) $\sum_{n=-\infty}^{\infty} |h(n)| = \infty$
    (c) $\sum_{n=-N}^{N} |h(n)| < \infty$
    (d) $\sum_{n=-N}^{N} |h(n)| = 0$ *Ans*: [a]

43. Represent the sequence $x(n) = \{1, 2, 3, 4\}$ as sum of impulses
    (a) $\delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3)$
    (b) $\delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)$
    (c) $\delta(n-1) + 2\delta(n) + 3\delta(n+1) + 4\delta(n+2)$
    (d) $\delta(n+1) + \delta(n) + \delta(n-1) + \delta(n-2)$ *Ans*: [b]

44. Response of an LTI systems is
    (a) Multiplication of input and impulse response
    (b) Subtraction of input and impulse response
    (c) Addition of input and impulse response
    (d) Convolution of input and impulse response
    Ans: [d]

45. Discrete-time systems are described by
    (a) Differential equations
    (b) Difference equations
    (c) Linear equations with variable coefficients
    (d) None *Ans*: [b]

46. Continuous-time systems are described by
    (a) Differential equations
    (b) Difference equations
    (c) Linear equations with variable coefficient
    (d) None *Ans*: [a]
47. Solution of difference equations can be obtained by
    (a) Laplace transform (b) Fourier transform
    (c) Z-transform (d) None *Ans*: [c]
48. Solution of differential equations can be obtained by
    (a) Laplace transform (b) Fourier transform
    (c) Z-transform (d) None *Ans*: [a]
49. Relation between system function and impulse response is
    (a) $H(z) = h(n)$
    (b) $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$
    (c) H (z) = Laplace Transform of h (n)
    (d) None *Ans*: [b]
50. Finite Geometric series $\sum_{n=0}^{N-1} a^n$ is
    (a) $\frac{1-a^{N-1}}{1-a}$ (b) $\frac{1-a^N}{1-a}$
    (c) $\frac{1-a^{N+1}}{1-a}$ (d) $\frac{1-a}{1-a^N}$ *Ans*: [b]

MODULE-1: DISCRETE FOURIER TRANSFORMS (DFT)

To perform frequency analysis on a discrete time sequence $x(n)$, we convert time-domain sequence to an equivalent frequency domain representation.

Applying Fourier Tranform on $x(n)$, we get $X(\omega)$, which is continuous and periodic function of frequency. It is not a computationally convenient representation of the sequence $x(n)$.

Representation of a sequence $x(n)$ by samples of its spectrum $X(\omega)$ is known as the Discrete Fourier Transform (DFT).

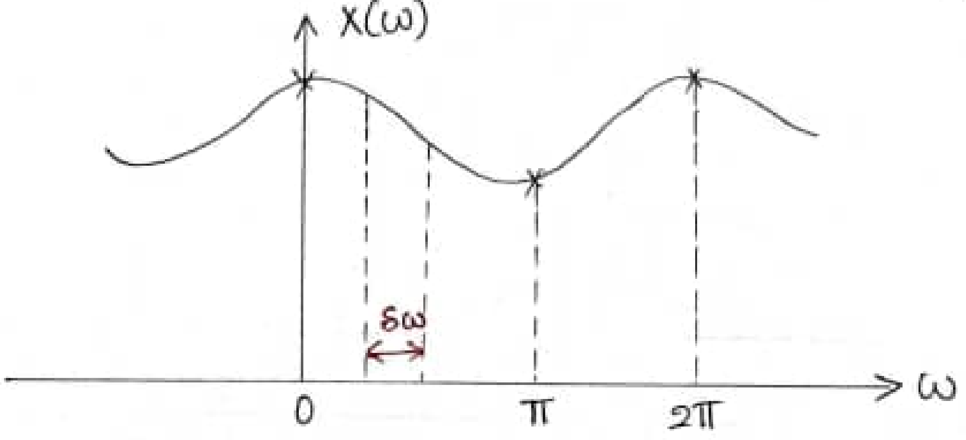
FREQUENCY DOMAIN SAMPLING :-

W.K.T aperiodic finite energy signals have continuous spectra.

Fourier Transform (FT) of aperiodic DTS $x(n)$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \text{———} (1)$$

Since $X(\omega)$ is periodic, with period 2π, only samples in the fundamental frequency range are obtained by sampling periodically in frequency at a spacing of $\delta\omega$ radians between successive samples. We take N equidistant samples in the interval $0 \le \omega \le 2\pi$ with spacing $\delta\omega = \frac{2\pi}{N}$.



Substituting $\omega = \frac{2\pi}{N}k$ in eqn ①,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{——} ②$$

where $k = 0, 1, 2, \ldots, N-1$

Summation in eqn ② can be subdivided into infinite number of summations, where each sum contains N terms.

$$\therefore X\left(\frac{2\pi}{N}k\right) = \cdots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=N}^{2N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \cdots$$

$$= \sum_{l=-\infty}^{+\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

If we change the index in the inner summation from n to $n-lN$ and interchange the order of summation, we get,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j\frac{2\pi}{N}k(n-lN)}$$

for $k=0,1,2,\ldots,N-1$.

But $e^{-j\frac{2\pi}{N}k(n-lN)} = e^{-j\frac{2\pi}{N}kn} \cdot e^{+j\frac{2\pi}{N}klN}$

where $e^{j\frac{2\pi}{N}klN} = e^{j2\pi kl} = 1$

$$\therefore X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{+\infty} x(n-lN)\right] e^{-j\frac{2\pi}{N}kn} \quad \text{——③}$$

$$\text{Let } x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \quad \text{——④}$$

$x_p(n)$ is obtained by the periodic repetition of $x(n)$ every N samples. It is periodic with fundamental period N.

$x_p(n)$ can hence be expanded in a Fourier Series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{+j\frac{2\pi}{N}kn} \quad \text{——⑤}$$

$n=0,1,2,\ldots,N-1$

with Fourier coefficients,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{——} ⑥$$

$$k = 0, 1, \ldots, N-1$$

Comparing equ's ③ & ⑥,

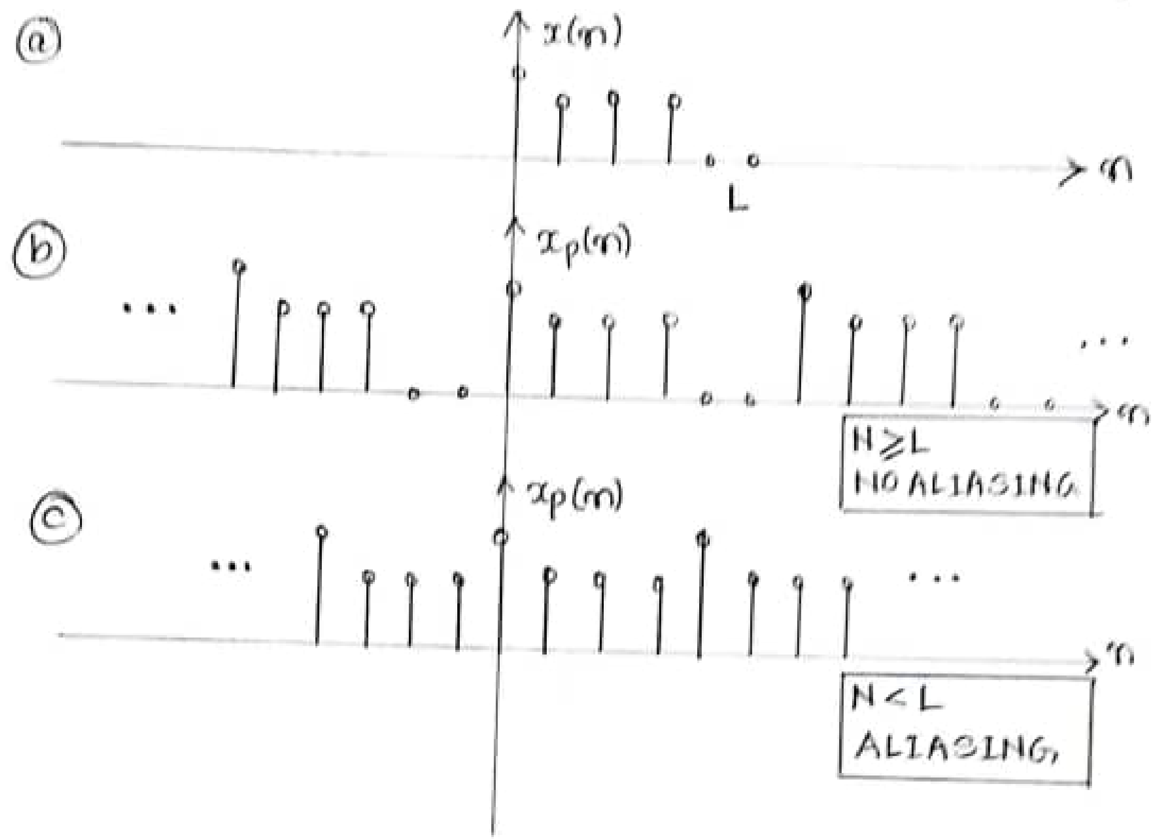
$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) \quad ; k = 0, 1, \ldots N-1 \quad \text{——} ⑦$$

Substituting equ ⑦ in equ ⑤,

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn} \quad \text{——} ⑧$$

Equation ⑧ provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$. However, to recover $X(\omega)$ or $x(n)$ from the samples certain condition needs to be satisfied.

Since $x_p(n)$ is the periodic extension of $x(n)$, it is clear that $x(n)$ can be recovered from $x_p(n)$ if there is no aliasing in the time domain i.e, if $x(n)$ is time limited to less than the period N of $x_p(n)$.



Consider fig (a), $x(n)$ is a sequence nonzero in the interval $0 \le n \le L-1$.

When $N \ge L$, $x(n) = x_p(n)$ for $0 \le n \le N-1$ [fig (b)]

Hence $x(n)$ can be recovered from $x_p(n)$.

When $N < L$, it cannot be recovered due to time-domain aliasing.

THE DISCRETE FOURIER TRANSFORM (DFT)

In general, the equally spaced frequency samples $X\left(\frac{2\pi}{N}k\right)$, $k = 0, 1, \ldots, N-1$, do not uniquely represent the original sequence $x(n)$ when $x(n)$ has infinite duration.

Instead, the frequency samples $X\left(\frac{2\pi}{N}k\right)$ correspond to a periodic sequence $x_p(n)$ of period N, where $x_p(n)$ is an aliased version of $x(n)$ given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

When the sequence $x(n)$ has a finite duration of length $L \leq N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$, where $x_p(n)$ over a single period is given by

$$x_p(n) = \begin{cases} x(n), & 0 \leq n \leq L-1 \\ 0, & L \leq n \leq N-1 \end{cases}$$

Hence, the frequency samples $X\left(\frac{2\pi k}{N}\right)$, $k = 0, 1, \ldots, N$ uniquely represent the finite duration sequence $x(n)$. Since $x(n) \equiv x_p(n)$ over a single period (padded by N−L zeros), the original finite duration sequence $x(n)$ can be obtained from the frequency samples $X\left(\frac{2\pi k}{N}\right)$ from Eqn ⑧.

⇒ Summarising: A finite-duration sequence $x(n)$ of length L has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n)\, e^{-j\omega n}, \quad 0 \leq \omega \leq 2\pi$$

Limits indicate $x(n)$ is zero outside the range $0 \leq n \leq L-1$.

When we sample $X(\omega)$ at equally spaced frequencies $\omega_k = \frac{2\pi k}{N}$, $k = 0, 1, \ldots, N-1$ where $N \geq L$, the samples are

$$X(k) \equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

or

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}}, \quad k = 0, 1, 2, \ldots, N-1$$

Upper index in the sum has been increased from $L-1$ to $N-1$ since $x(n) = 0$ for $n \geq L$.

The above relation is a formula for transforming a sequence $\{x(n)\}$ of length $L \leq N$ into a sequence of frequency samples $\{X(k)\}$ of length N.

Since, the frequency samples are obtained by evaluating the Fourier transform $X(\omega)$ at a set of N (equally spaced) discrete frequencies, the above relation is called discrete Fourier transform (DFT) of $x(n)$.

Let $W_N = e^{-j\frac{2\pi}{N}}$, then

$$\therefore \quad \boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}}, \quad k = 0, 1, 2, \ldots, N-1$$

The relation which allows us to recover the sequence $x(n)$ from the frequency samples

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn}, \quad n = 0, 1, 2, \ldots, N-1$$

$$\text{or } x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

is called the Inverse DFT (IDFT).

DFT AS A LINEAR TRANFORMATION

WKT,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$k = 0, 1, 2, \ldots, N-1$$

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$n = 0, 1, 2, \ldots, N-1$$

where $W_N = e^{-j\frac{2\pi}{N}}$ Twiddle Factor

From the above equations we observe that the computation of each point of the DFT requires N-complex multiplications and $(N-1)$ complex additions.

Hence, N-point DFT requires N^2 complex multiplications and $N(N-1)$ complex additions.

DFT and IDFT can be viewed as linear transformations on sequences $\{x(n)\}$ and $\{X(k)\}$, respectively.

Let us define a N-point vector x_N of the signal sequence $x(n)$, an N-point vector X_N of frequency samples and a $N \times N$ matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} W_N^{0.0} & W_N^{0.1} & \cdots & W_N^{0(N-1)} \\ W_N^{1.0} & W_N^{1.1} & \cdots & W_N^{1(N-1)} \\ \vdots & \vdots & & \\ W_N^{(N-1)0} & W_N^{(N-1).1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\text{or } W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

From the above definitions, the N-point DFT can be expressed in matrix form as

$$\boxed{X_N = W_N x_N}$$

Where W_N is the matrix of the linear transformation. We observe that W_N is a symmetric matrix. If we assume that inverse of W_N exists then IDFT can be expressed in matrix form as

$$x_N = W_N^{-1} X_N$$

$$\text{or} \quad \boxed{x_N = \frac{1}{N} W_N^{*} X_N}$$

where W_N^* denotes the complex conjugate of W_N.

PROBLEMS

1) Compute the 4-point DFT of a sequence $x(n) = \{1, 2, 3, 4\}$.

Soln:— I- METHOD :

From the defn of DFT wkt,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ; k = 0, 1, .., N-1$$

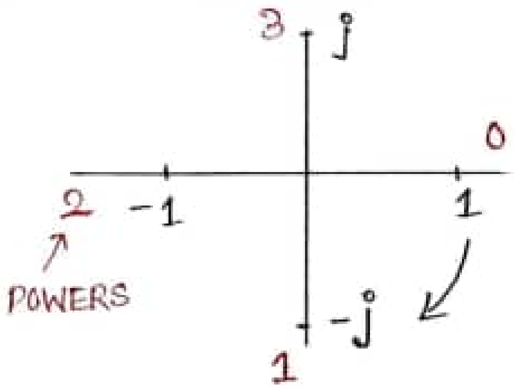
Given $N=4$,

$$\therefore X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}$$

$$= x(0) + x(1)\,W_4^{k} + x(2)\,W_4^{2k} + x(3)\,W_4^{3k}$$

$$\boxed{X(k) = 1 + 2W_4^{k} + 3W_4^{2k} + 4W_4^{3k}} \quad ; k = 0, 1, 2, 3$$

Twiddle Factors:



For $N=4$, First four twiddle factors.

$W_4^0 = 1 \qquad W_4^2 = -1$

$W_4^1 = -j \qquad W_4^3 = j$

Alternatively, use $W_N = e^{-j\frac{2\pi}{N}}$ and solve.

Let $k=0$, $X(0) = 1 + 2 + 3 + 4 = \underline{\underline{10}}$

Let $k=1$, $X(1) = 1 + 2W_4^1 + 3W_4^2 + 4W_4^3$

$= 1 + 2(-j) + 3(-1) + 4(j)$

$= \underline{\underline{-2 + 2j}}$

Let $k=2$, $X(2) = 1 + 2W_4^2 + 3W_4^4 + 4W_4^6$

$= 1 + 2(-1) + 3(1) + 4(-1)$

$= \underline{\underline{-2}}$

Let $k=3$, $X(3) = 1 + 2W_4^3 + 3W_4^6 + 4W_4^9$

$= 1 + 2(j) + 3(-1) + 4(-j)$

$= \underline{\underline{-2 - 2j}}$

$$\therefore \boxed{X(k) = \{10, -2+2j, -2, -2-2j\}}$$

II – METHOD :-

$$X_N = x_N W_N \qquad X_N \text{ \& } x_N \rightarrow \text{Row vectors}$$

For $N=4$, $X_4 = x_4 W_4$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

From periodicity property of W_N : $W_N^{K+N} = W_N^K$

and symmetry property : $W_N^{K+\frac{N}{2}} = -W_N^K$

$$X_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\therefore \boxed{X(K) = \{ 10, -2+2j, -2, -2-2j \}}$$

Additional problems on DFT :-

2) $x(n) = \{0, 1, 2, 3\}$ for $N=4$.

ANS: $X(K) = \{ 6, -2+2j, -2, -2-2j \}$

3) $x(n) = \{ 1, 2, 2, 1 \}$ for $N=4$.

ANS: $X(K) = \{ 6, -1-j, 0, -1+j \}$

4) $x(n) = \delta(n) + \delta(n-1) - \delta(n-2) - \delta(n-3)$

$= \{1, 1, -1, -1\}$

ANS : $X(k) = \{0, 2-2j, 0, 2+2j\}$

5) Find the 4-point IDFT of $X(k) = \{10, -2+2j, -2, -2-2j\}$.

Soln:- From the defn of IDFT,

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$n = 0, 1, 2, \ldots, N-1$

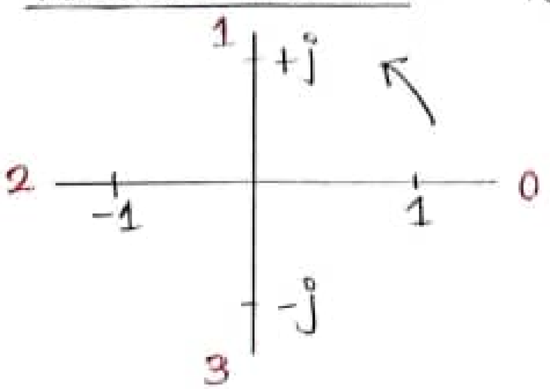
For $N=4$,

$$x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, W_4^{-kn}$$

$$x(n) = \frac{1}{4}\left[X(0) + X(1)W_4^{-n} + X(2)W_4^{-2n} + X(3)W_4^{-3n}\right]$$

$$x(n) = \frac{1}{4}\left[10 + (-2+2j)W_4^{-n} + (-2)W_4^{-2n} + (-2-2j\,W_4^{-3n}\right], \quad n = 0, 1, 2, 3.$$

TWIDDLE FACTORS:



For $N=4$,

$W_4^{0} = 1$ $\qquad W_4^{-2} = -1$

$W_4^{-1} = j$ $\qquad W_4^{-3} = -j$

Let $n=0$,

$$x(0) = \frac{1}{4}\left[10 - 2 + 2j - 2 - 2 - 2j\right]$$

$$x(0) = \frac{4}{4} = \underline{\underline{1}}$$

Let $n=1$,

$$x(1) = \frac{1}{4}\left[10 + (-2+2j)W_4^{-1} + (-2)W_4^{-2} + (-2-2j)W_4^{-3}\right]$$

$$= \frac{1}{4}\left[10 + (-2+2j)(j) + (-2)(-1) + (-2-2j)(-j)\right]$$

$$x(1) = \frac{8}{4} = \underline{\underline{2}}$$

Let $n=2$,

$$x(2) = \frac{1}{4}\left[10 + (-2+2j)W_4^{-2} + (-2)W_4^{-4} + (-2-2j)W_4^{-6}\right]$$

$$= \frac{1}{4}\left[10 + (-2+2j)(-1) + (-2)(1) + (-2-2j)(-1)\right]$$

$$x(2) = \frac{12}{4} = \underline{\underline{3}}$$

Let $n=3$, $x(3) = \frac{1}{4}\left[10 + (-2+2j)W_4^{-3} + (-2)W_4^{-6} + (-2-2j)W_4^{-9}\right]$

$$x(3) = \frac{16}{4} = \underline{\underline{4}}$$

$\therefore$ $\boxed{x(n) = \{1, 2, 3, 4\}}$

II - METHOD :-

WKT, $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$

Above equation can also be written as,

$$x_N = \frac{1}{N} X_N W_N^*$$ $x_N, X_N \rightarrow$ Row vectors.

Given $N=4$,

$$x_4 = \frac{1}{4} X_4 W_4^*$$

$$x_4 = \frac{1}{4}\begin{bmatrix} 10 & (-2+2j) & -2 & (-2-2j) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 10-2+2j-2-2-2j \\ 10-2j-2+2+2j-2 \\ 10+2-2j-2+2+2j \\ 10+2j+2+2-2j+2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 12 \\ 16 \end{bmatrix}$$

$\therefore$ $x(n) = \{1, 2, 3, 4\}$

NOTE: ANSWER IS IN ROW, REPRESENTED AS COLUMN ONLY FOR CONVENIENCE.

⑥ Find the DFT of $x(n) = \delta(n)$.

Soln :- From defn of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Hence, $$DFT\{\delta(n)\} = \sum_{n=0}^{N-1} \delta(n)\, W_N^{kn}$$

W.K.T, $$\delta(n) = \begin{cases} 1, & \text{at } n=0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

$$\therefore DFT\{\delta(n)\} = \delta(0)\, W_N^{0} = 1 \;//$$

or $$\boxed{\delta(n) \overset{N\text{-PT DFT}}{\longleftrightarrow} 1}$$

⑦ Find the DFT of $x(n) = \delta(n - n_0)$.

Soln :- From the defn of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Hence, $$DFT\{\delta(n-n_0)\} = \sum_{n=0}^{N-1} \delta(n-n_0)\, W_N^{kn}$$

W.K.T, $$\delta(n-n_0) = \begin{cases} 1, & \text{at } n=n_0 \\ 0, & \text{for } n \neq n_0 \end{cases}$$

$$\therefore DFT\{\delta(n-n_0)\} = \delta(0)\, W_N^{kn_0} = W_N^{kn_0}$$

$$\boxed{\delta(n-n_0) \overset{DFT}{\longleftrightarrow} W_N^{kn_0}}$$

⑧ Find N-point DFT of $x(n) = e^{\frac{j2\pi}{N}k_0 n}$.

Soln:— From definition of DFT

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Hence, $$DFT\left\{e^{\frac{j2\pi}{N}k_0 n}\right\} = \sum_{n=0}^{N-1} e^{\frac{j2\pi}{N}k_0 n} \cdot e^{-\frac{j2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left[e^{-\frac{j2\pi}{N}(k-k_0)} \right]^n$$

$$X(k) = \frac{1 - \left[e^{-\frac{j2\pi}{N}(k-k_0)} \right]^N}{1 - e^{-\frac{j2\pi}{N}(k-k_0)}} \qquad \because \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

at $k = k_0$, $X(k) = \frac{0}{0}$, Using L-Hospital's Method,

$$= \left. \frac{0 - \left(-j2\pi\, e^{-j2\pi(k-k_0)}\right)}{0 - \left(-\frac{j2\pi}{N}\, e^{-\frac{j2\pi}{N}(k-k_0)}\right)} \right|_{k=k_0}$$

At $k = k_0$,

$$= \frac{j2\pi \cdot e^0}{\frac{j2\pi}{N} \cdot e^0} = N$$

$\therefore X(k) = N$ at $k = k_0$

at $k \neq k_0$ $X(k) = 0$

$\therefore$ $\boxed{X(k) = N\,\delta(k-k_0)}$ or $\boxed{X(k) = \begin{cases} N & \text{at } k = k_0 \\ 0 & \text{for } k \neq k_0 \end{cases}}$

⑨ Find the DFT of $x(n) = e^{-j\frac{2\pi}{N}k_0 n}$.

Soln:- From the defn of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Hence, $$DFT\left\{e^{-j\frac{2\pi}{N}k_0 n}\right\} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}k_0 n} \cdot e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left[e^{-j\frac{2\pi}{N}(k+k_0)}\right]^n$$

$$= \frac{1 - \left[e^{-j\frac{2\pi}{N}(k+k_0)}\right]^N}{1 - e^{-j\frac{2\pi}{N}(k+k_0)}} \qquad \because \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

at $k = -k_0$, $X(k) = \frac{0}{0}$.

Using L-Hospital's Rule,

$$X(k) = \left.\frac{0 - (-j2\pi)\, e^{-j2\pi(k+k_0)}}{0 - \left(-j\frac{2\pi}{N}\right) e^{-j\frac{2\pi}{N}(k+k_0)}}\right|_{k=-k_0}$$

at $k = -k_0$, $X(k) = \frac{j2\pi}{j2\pi/N} = N$

at $k \neq -k_0$, $X(k) = 0$

$\therefore$ $\boxed{X(k) = N\,\delta(k+k_0)}$ or $\boxed{X(k) = \begin{cases} N & \text{at } k=k_0 \\ 0 & \text{at } k\neq k_0 \end{cases}}$

or $X(k) = N\,\delta(k - N + k_0)$

(10) Find the N-point DFT of $x(n) = \cos \frac{2\pi}{N} k_0 n$

Sol'n:- From def'n of DFT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$DFT\left\{\cos\left(\frac{2\pi}{N}k_0 n\right)\right\} = \sum_{n=0}^{N-1} \cos\left(\frac{2\pi}{N}k_0 n\right) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left[\frac{e^{j\frac{2\pi}{N}k_0 n} + e^{-j\frac{2\pi}{N}k_0 n}}{2}\right] e^{-j\frac{2\pi}{N}kn}$$

$\because \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$$= \frac{1}{2}\left\{\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} + \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+k_0)n}\right\}$$

$$= \frac{1}{2}\left\{N\delta(k-k_0) + N\delta(k+k_0)\right\}$$

$$\therefore \boxed{\cos\left(\frac{2\pi}{N}k_0 n\right) \xleftrightarrow{DFT} \frac{N}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right]}$$

(11) Similar problem, $x(n) = \sin \frac{2\pi}{N} k_0 n$

ANS: $\sin\left(\frac{2\pi}{N}k_0 n\right) \xleftrightarrow{DFT} \frac{N}{2j}\left[\delta(k-k_0) - \delta(k+k_0)\right]$

(12) Find the 5-point DFT of $x(n) = \{1, 1, 1\}$.

Soln: From the defn of DFT, WKT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$k = 0, 1, \ldots, N-1$

Given $N = 5$,

$$X(k) = \sum_{n=0}^{4} x(n) W_5^{kn} \quad ; k = 0, 1, 2, 3, 4$$

$$X(k) = x(0) + x(1) W_5^{k} + x(2) W_5^{2k} + x(3) W_5^{3k} + x(4) W_5^{4k}$$

$$\boxed{X(k) = 1 + W_5^{k} + W_5^{2k}} \quad k = 0, 1, 2, 3, 4$$

Twiddle Factors: $N = 5$

$W_5^0 = 1$

$$W_5^1 = e^{-j\frac{2\pi}{5}} = \cos\left(\frac{2\pi}{5}\right) - j\sin\left(\frac{2\pi}{5}\right)$$

$$= 0.3090 - j0.95$$

$$W_5^2 = e^{-j\frac{2\pi}{5}\cdot 2} = \cos\left(\frac{4\pi}{5}\right) - j\sin\left(\frac{4\pi}{5}\right)$$

$$= -0.809 - j0.5878$$

$$W_5^3 = e^{-j\frac{2\pi}{5}\cdot 3} = \cos\left(\frac{6\pi}{5}\right) - j\sin\left(\frac{6\pi}{5}\right)$$

$$= -0.809 + j0.587$$

$$W_5^4 = e^{-j\frac{2\pi}{5}\cdot 4} = \cos\left(\frac{8\pi}{5}\right) - j\sin\left(\frac{8\pi}{5}\right)$$

$$= 0.309 + j\,0.95$$

Evaluate $X(k)$,

at $k=0$, $X(0) = 1 + 1 + 1 = \underline{\underline{3}}$

at $k=1$, $X(1) = 1 + W_5^1 + W_5^2 = \underline{\underline{0.5 - j1.5388}}$

at $k=2$, $X(2) = 1 + W_5^2 + W_5^4 = \underline{\underline{0.5 + j0.3633}}$

at $k=3$, $X(3) = 1 + W_5^3 + W_5^6 = \underline{\underline{0.5 - j0.3633}}$

at $k=4$, $X(4) = 1 + W_5^4 + W_5^8 = \underline{\underline{0.5 + j1.5388}}$

$$X(k) = \{3,\ 0.5 - j1.5388,\ 0.5 + j0.3633,\ 0.5 - j0.3633,\ 0.5 + j1.5388\}$$

(13) Find IDFT for the following sequence

$$X(k) = \{5,\ 0,\ (1-j),\ 0,\ 1,\ 0,\ (1+j),\ 0\}$$

Soln:- Length of the sequence is 8.

$\therefore N = 8$

From the defn of IDFT,

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

where $n = 0, 1, 2, \ldots, N-1$.

For $N=8$,

$$x(n) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, W_8^{-kn}$$

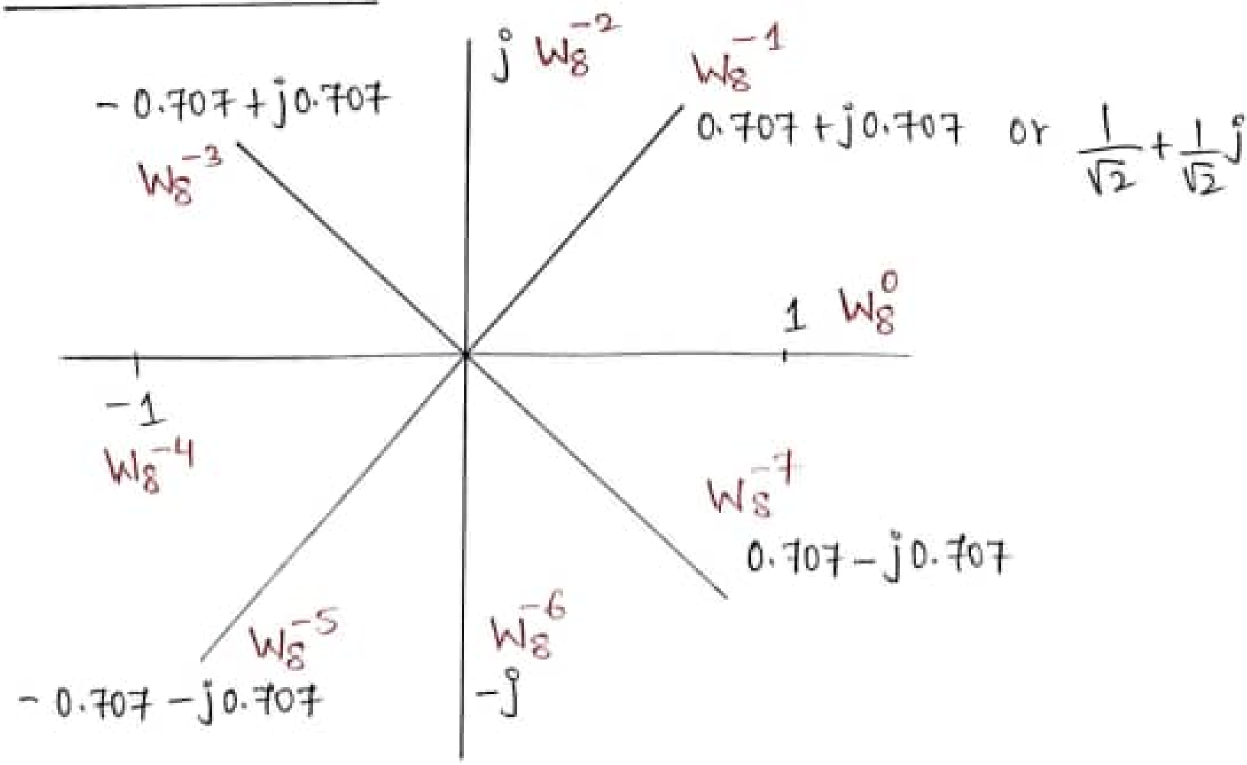
$$x(n) = \frac{1}{8}\Big[X(0)W_8^0 + X(1)W_8^{-n} + X(2)W_8^{-2n} + X(3)W_8^{-3n} + X(4)W_8^{-4n} + X(5)W_8^{-5n} + X(6)W_8^{-6n} + X(7)W_8^{-7n} \Big]$$

$$x(n) = \frac{1}{8}\Big[5 + 0 + (1-j)W_8^{-2n} + 0 + W_8^{-4n} + 0 + (1+j)W_8^{-6n} + 0 \Big]$$

where $n = 0, 1, 2, \ldots, 7$

or $$\boxed{x(n) = \frac{1}{8}\Big[5 + (1-j)W_8^{-2n} + W_8^{-4n} + (1+j)W_8^{-6n} \Big]}$$

TWIDDLE FACTORS : $N=8$



For $n=0$, $x(0) = \frac{1}{8}\left\{ 5 + 1-j + 1 + 1+j \right\}$

$$= \frac{8}{8} = \underline{\underline{1}}$$

For $n=1$, $x(1) = \frac{1}{8}\left\{ 5 + (1-j)W_8^{-2} + W_8^{-4} + (1+j)W_8^{-6} \right\}$

$$= \frac{1}{8}\left\{ 5 + (1-j)j + (-1) + (1+j)(-j) \right\}$$

$$= \frac{1}{8}\left\{ 5 + j - j^2 - 1 - j - j^2 \right\}$$

$$= \frac{6}{8} = \underline{\underline{\frac{3}{4}}}$$

For $n=2$, $x(2) = \frac{1}{8}\left\{ 5 + (1-j)W_8^{-4} + W_8^{-8} + (1+j)W_8^{-12} \right\}$

$$= \frac{1}{8}\left\{ 5 + (1-j)(-1) + 1 + (1+j)(-1) \right\}$$

$$= \frac{1}{8}\left\{ 5 - 1 + j + 1 - 1 - j \right\}$$

$$= \frac{4}{8} = \underline{\underline{\frac{1}{2}}}$$

For $n=3$, $x(3) = \frac{1}{8}\left\{ 5 + (1-j)W_8^{-6} + W_8^{-12} + (1+j)W_8^{-18} \right\}$

$$= \frac{1}{8}\left\{ 5 + (1-j)(-j) + (-1) + (1+j)(j) \right\}$$

$$x(3) = \frac{1}{8}\left\{5 - j + j^2 - 1 + j + j^2\right\}$$

$$= \frac{1}{8}\left\{5 - j - 1 - 1 + j - 1\right\} = \frac{2}{8} = \underline{\underline{\frac{1}{4}}}$$

for $n=4$, $$x(4) = \frac{1}{8}\left\{5 + (1-j)\,W_8^{-8} + W_8^{-16} + (1+j)\,W_8^{-24}\right\}$$

$$= \frac{1}{8}\left\{5 + (1-j) + 1 + (1+j)\right\}$$

$$= \frac{8}{8} = \underline{\underline{1}}$$

for $n=5$, $$x(5) = \frac{1}{8}\left\{5 + (1-j)\,W_8^{-10} + W_8^{-20} + (1+j)\,W_8^{-30}\right\}$$

$$= \frac{1}{8}\left\{5 + (1-j)j + (-1) + (1+j)(-j)\right\}$$

$$= \frac{1}{8}\left\{5 + j - j^2 - 1 - j - j^2\right\}$$

$$= \frac{6}{8} = \underline{\underline{\frac{3}{4}}}$$

for $n=6$, $$x(6) = \frac{1}{8}\left\{5 + (1-j)\,W_8^{-12} + W_8^{-24} + (1+j)\,W_8^{-36}\right\}$$

$$= \frac{1}{8}\left\{5 + (1-j)(-1) + 1 + (1+j)(-1)\right\}$$

$$= \frac{1}{8}\left\{5 - 1 + j + 1 - 1 - j\right\} = \frac{4}{8} = \underline{\underline{\frac{1}{2}}}$$

for $n=7$, $x(7) = \frac{1}{8}\left\{ 5 + (1-j)\,W_8^{-14} + W_8^{-28} + (1+j)\,W_8^{-42} \right\}$

$= \frac{1}{8}\left\{ 5 + (1-j)(-j) + (-1) + (1+j)(j) \right\}$

$= \frac{1}{8}\left\{ 5 - j + j^2 - 1 + j + j^2 \right\}$

$= \frac{2}{8} = \frac{1}{4}$

$\therefore \; x(n) = \left\{ 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 1, \frac{3}{4}, \frac{1}{2}, \frac{1}{4} \right\}$

(14) Find DFT of the sequence $x(n) = \begin{cases} 1, & 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$

for $N=8$. Plot $|X(k)|$ & $\angle X(k)$.

Soln:- From the defn of DFT,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

for $N=8$,

$$X(k) = \sum_{n=0}^{7} x(n)\, W_8^{kn} \quad , 0 \le k \le 7$$

Given $x(n) = \{1, 1, 1\}$

$$\therefore X(k) = 1 + W_8^{k} + W_8^{2k} + 0 + 0 + 0 + 0 + 0$$

$$\boxed{X(k) = 1 + W_8^{k} + W_8^{2k}}, \quad k = 0, 1, 2, \ldots, 7$$

Twiddle Factors for $N = 8$

$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$

$W_8^4 = -1$

$W_8^5 = -0.707 + j0.707$

$W_8^6 = j$

$W_8^7 = 0.707 + j0.707$

(Locate points on the unit circle in complex plane direction is clockwise for DFT)

for $k=0$, $X(0) = 1 + 1 + 1 = \underline{\underline{3}}$

for $k=1$, $X(1) = 1 + W_8^1 + W_8^2$

$= 1 + (0.707 - j0.707) + (-j)$

$= \underline{\underline{1.707 - j1.707}}$

for $k=2$, $X(2) = 1 + W_8^2 + W_8^4$

$= 1 + (-j) + (-1) = \underline{\underline{-j}}$

for $k=3$, $X(3) = 1 + W_8^3 + W_8^6$

$= 1 + (-0.707 - j0.707) + j$

$= \underline{\underline{0.293 + j0.293}}$

for $k=4$, $X(4) = 1 + W_8^4 + W_8^8$

$= 1 + (-1) + 1 = \underline{\underline{1}}$

for $k=5$, $X(5) = 1 + W_8^5 + W_8^{10}$

$= 1 + (-0.707 + j0.707) + (-j)$

$= \underline{\underline{0.293 - j0.293}}$

for $k=6$, $X(6) = 1 + W_8^6 + W_8^{12}$

$= 1 + j + (-1) = \underline{\underline{j}}$

for $k=7$, $X(7) = 1 + W_8^7 + W_8^{14}$

$= 1 + 0.707 + j0.707 + j$

$= \underline{\underline{1.707 + j1.707}}$

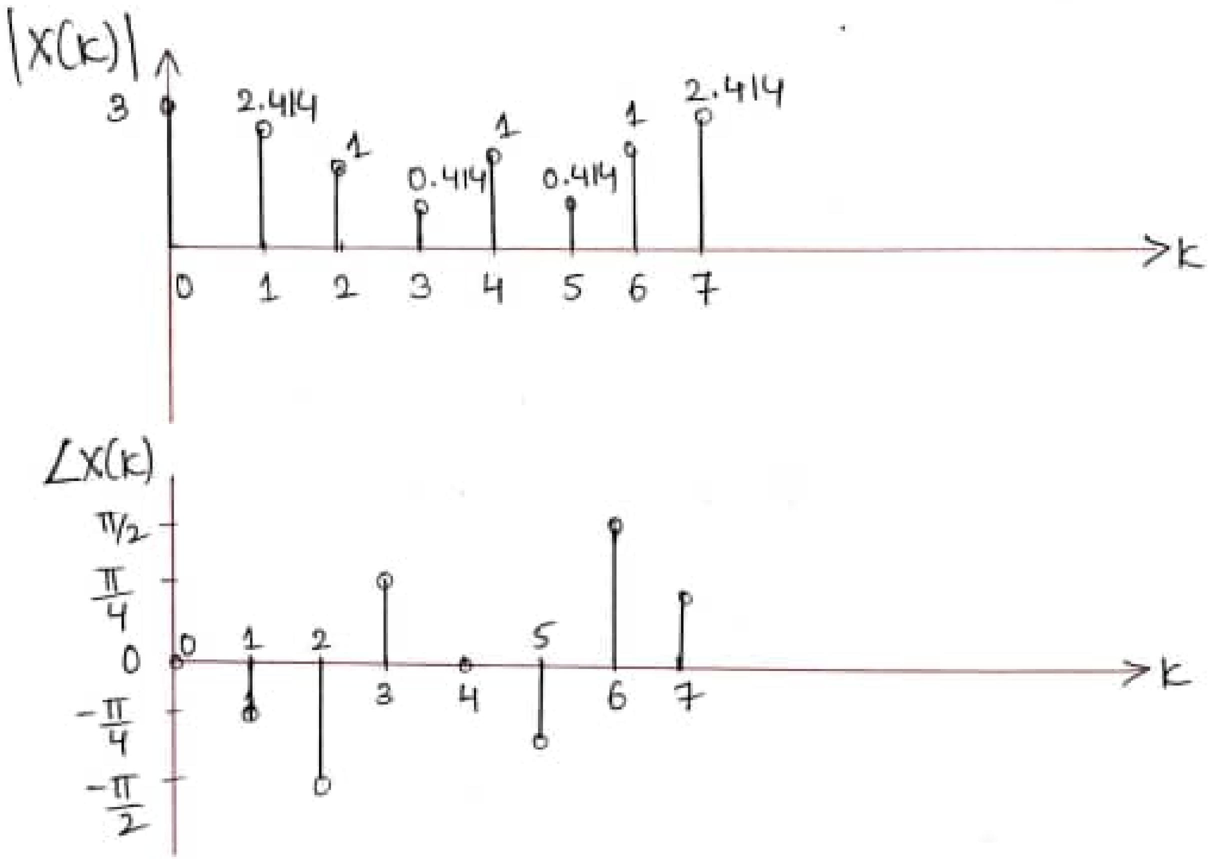
$\therefore X(k) = \{3, (1.707 - j1.707), -j, (0.293 + j0.293), 1, (0.293 - j0.293), j, (1.707 + j1.707)\}$

For magnitude Plot,

$|X(k)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$

Phase Plot,

$\angle X(k) = \{0, -\frac{\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\}$


|X(k)|
3
2.414
1
0.414
1
0.414
1
2.414
k
0 1 2 3 4 5 6 7
∠X(k)
π/2
π/4
0
−π/4
−π/2
0 1 2 3 4 5 6 7
k

PROPERTIES OF DFT

1) LINEARITY :-

STATEMENT:

If $x_1(n) \xleftrightarrow{DFT} X_1(k)$

$x_2(n) \xleftrightarrow{DFT} X_2(k)$

then $a x_1(n) + b x_2(n) \xleftrightarrow{DFT} a X_1(k) + b X_2(k)$

Proof: W.K.T., $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

Let $x(n) = x_1(n)$,

then $X_1(k) = \sum_{n=0}^{N-1} x_1(n) W_N^{kn}$ —— (1)

Let $x(n) = x_2(n)$,

then $X_2(k) = \sum_{n=0}^{N-1} x_2(n) W_N^{kn}$ —— (2)

Let $x(n) = a x_1(n) + b x_2(n)$, then

$$DFT\{a x_1(n) + b x_2(n)\} = \sum_{n=0}^{N-1} [a x_1(n) + b x_2(n)] W_N^{kn}$$

$$= a \sum_{n=0}^{N-1} x_1(n) W_N^{kn} + b \sum_{n=0}^{N-1} x_2(n) W_N^{kn}$$

$$= a X_1(k) + b X_2(k)$$

Hence the proof.

2) PERIODICITY

STATEMENT: If $x(n) \xleftrightarrow{DFT} X(k)$

then $x(n+N) = x(n)$

$X(k+N) = X(k)$

or $x(n+N) \xleftrightarrow{DFT} X(k)$

Proof: a) TPT $x(n+N) = x(n)$

W.K.T $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$

replace n by $n+N$,

$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-k(n+N)}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \cdot W_N^{-kN}$$

W.K.T $W_N^{-kN} = 1$

$$\therefore x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$\boxed{x(n+N) = x(n)}$$

Hence proved.

b) T.P.T $X(k+N) = X(k)$

WKT $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

replace k by k+N,

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{(k+N)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \cdot W_N^{Nn}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \qquad \because W_N^{Nn} = 1$$

$$\therefore \boxed{X(k+N) = X(k)}$$

Hence proved.

c) T.P.T $x(n+N) \xleftrightarrow{DFT} X(k)$

WKT, $DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

Let $x(n) = x(n+N)$, then

$$DFT\{x(n+N)\} = \sum_{n=0}^{N-1} x(n+N)\, W_N^{kn}$$

Put $l = n+N \Rightarrow n = l-N$

limits : $n = 0$, $l = N$

$n = N-1$, $l = N-1+N = 2N-1$

$$= \sum_{l=N}^{2N-1} x(l)\, W_N^{k(l-N)}$$

$$= \sum_{l=N}^{2N-1} x(l)\, W_N^{kl} \cdot W_N^{-kN}$$

Since $W_N^{-kN} = 1$ & limits N to $2N-1$ can be replaced by 0 to $N-1$ using periodicity.

$$\text{DFT}\{x(n+N)\} = \sum_{l=0}^{N-1} x(l)\, W_N^{kl}$$

$$\therefore \boxed{\text{DFT}\{x(n+N)\} = X(k)}$$

Hence proved.

③ CIRCULAR TIME-SHIFT

Circular time shift operation on an N-point sequence $x(n)$ is given by $x((n-m))_{\text{mod } N}$, $x((n-m))_N$ or $x(n-m, \text{mod } N)$.

STATEMENT: If $x(n) \xleftrightarrow{DFT} X(k)$

then $x((n-m))_N \xleftrightarrow{DFT} X(k)\, e^{-j\frac{2\pi}{N}km}$

or $x((n-m))_N \xleftrightarrow{DFT} X(k)\, W_N^{km}$

|||ly $x((n+m))_N \xleftrightarrow{DFT} X(k)\, e^{j\frac{2\pi}{N}km}$

or $x((n+m))_N \xleftrightarrow{DFT} X(k)\, W_N^{-km}$

Proof: W.K.T $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

Let $x(n) = x((n-m))_N$

then $DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N \cdot W_N^{kn}$

Let $n-m = l \Rightarrow n = l+m$, limits:

at $n=0$, $l=-m$

$n=N-1$, $l=N-1-m$

$$DFT\{x((n-m))_N\} = \sum_{l=-m}^{N-1-m} x(l)\, W_N^{k(l+m)}$$

$$= \sum_{l=0}^{N-1} x(l)\, W_N^{kl} \cdot W_N^{km} \quad \text{using periodicity}$$

$$\therefore \boxed{DFT\{x((n-m))_N\} = X(k)\, W_N^{km}}$$

Example (15) If $x(n) = \{1, 2, 3, 4\}$ find $y(n)$, given $y(n) = x((n-3))_4$.

Soln:- Given $y(n) = x((n-3))_4$

at $n=0$, $y(0) = x((-3))_4 = x(4-3)$

$= x(1) = \underline{\underline{2}}$

at $n=1$, $y(1) = x((1-3))_4 = x((-2))_4$

$= x(4-2) = x(2) = \underline{\underline{3}}$

at $n=2$, $y(2) = x((2-3))_4 = x((-1))_4$

$= x(4-1) = x(3) = \underline{\underline{4}}$

at $n=3$, $y(3) = x((3-3))_4 = x(0)$

$= \underline{\underline{1}}$

$\therefore \boxed{y(n) = \{2, 3, 4, 1\}}$

(16) If $x(n) = \{1, 2, 2, 1\}$ find the DFT of $y(n) = x((n-2))_4$.

Soln:- First find $X(k)$.

$X_N = x_N W_N$

$$X(k) = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\therefore X(k) = \begin{bmatrix} 6 & -1-j & 0 & -1+j \end{bmatrix}$$

Given $y(n) = x((n-2))_4$

Taking DFT and applying circular time-shift property,

$$Y(k) = X(k)\, W_4^{2k} \quad , \quad k = 0, 1, 2, 3$$

at $k=0$, $Y(0) = X(0)\, W_4^{0} = (6)(1) = 6$

at $k=1$, $Y(1) = X(1)\, W_4^{2} = (-1-j)(-1) = 1+j$

at $k=2$, $Y(2) = X(2)\, W_4^{4} = 0$

at $k=3$, $Y(3) = X(3)\, W_4^{6} = (-1+j)(-1) = 1-j$

$$\therefore \boxed{Y(k) = \{6, 1+j, 0, 1-j\}}$$

4) TIME REVERSAL

STATEMENT: If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then. $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT}$

$$X((-k))_N = X(N-k)$$

Proof:- From defn of DFT, we have

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Hence, $$DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, W_N^{kn}$$

Put $m = N-n \Rightarrow n = N-m$

limits : at $n=0$, $m=N$

$n=N-1$, $m = N-(N-1) = 1$

$$\therefore DFT\{x(N-n)\} = \sum_{m=N}^{1} x(m)\, W_N^{k(N-m)}$$

$$= \sum_{m=0}^{N-1} x(m)\, W_N^{kN} \cdot W_N^{-km}$$ using periodicity property.

Also, $W_N^{kN} = 1$

$$DFT\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)\, W_N^{(-k)m}$$

or $$\boxed{DFT\{x(N-n)\} = X((-k))_N}$$

(17) Example : $x(n) = \{5, 2, -3, 7\}$. Find $y(n) = x((-n))_4$.

Sol'n: Consider $y(n) = x((-n))_4$

or $y(n) = x(4-n)$

at $n=0$, $y(0) = x(4) = x(0) = 5$

$n=1$, $y(1) = x(4-1) = x(3) = 7$

$n=2$, $y(2) = x(4-2) = x(2) = -3$

$n=3$, $y(3) = x(4-3) = x(1) = 2$

$\therefore$ $y(n) = \{5, 7, -3, 2\}$

(18) Given $x(n) = \{1, 2, 3, 4\}$ with 4-point DFT $X(k) = \{10, -2+2j, -2, -2-2j\}$. Find the 4-point DFT of the sequence $y(n) = \{1, 4, 3, 2\}$.

Sol'n:- Observing $x(n)$ & $y(n)$ we find that

$$y(n) = x((-n))_4$$

Taking DFT on both sides and applying time reversal property

$$Y(k) = X((-k))_4 = X(4-k)$$

at $k=0$, $Y(0) = X(0) = 10$

$k=1$, $Y(1) = X(4-1) = X(3) = -2-2j$

$k=2$, $Y(2) = X(4-2) = X(2) = -2$

$k=3$, $Y(3) = X(4-3) = X(1) = -2+2j$

$$\therefore \boxed{Y(k) = \{10, -2-2j, -2, -2+2j\}}$$

5) CIRCULAR FREQUENCY SHIFT

STATEMENT: If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then $x(n)\, e^{\frac{j2\pi}{N}mn} \xleftrightarrow[N]{DFT} X((k-m))_N$

or $x(n)\, W_N^{-mn} \xleftrightarrow{DFT} X((k-m))_N$

IIIly

$x(n)\, e^{-\frac{j2\pi}{N}mn} \xleftrightarrow{DFT} X((k+m))_N$

or $x(n)\, W_N^{mn} \xleftrightarrow{DFT} X((k+m))_N$

Proof:- $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

$$DFT\left\{x(n)\, e^{\frac{j2\pi}{N}mn}\right\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi}{N}mn} \cdot e^{-\frac{j2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi}{N}(k-m)n}$$

$$= X((k-m))_N$$

Hence the proof.

(19) Find N-point DFT of $x_1(n) = \cos\left(\frac{2\pi}{N}k_0 n\right) x(n)$.

Sol'n:-

Given $x_1(n) = \cos\left(\frac{2\pi}{N}k_0 n\right) x(n)$

$$= \left[\frac{e^{j\frac{2\pi}{N}k_0 n} + e^{-j\frac{2\pi}{N}k_0 n}}{2}\right] x(n)$$

If $x(n) \xleftrightarrow{DFT} X(k)$

Using circular frequency shift property,

$$e^{j\frac{2\pi}{N}k_0 n} x(n) \xleftrightarrow{DFT} X((k-k_0))_N$$

$$e^{-j\frac{2\pi}{N}k_0 n} x(n) \xleftrightarrow{DFT} X((k+k_0))_N$$

Hence DFT $\{x_1(n)\} = X_1(k)$

and $$X_1(k) = \frac{1}{2}\left[X((k-k_0))_N + X((k+k_0))_N\right]$$

(20) Find N-point DFT of $x_2(n) = \sin\left(\frac{2\pi}{N}k_0 n\right) x(n)$

ANS : $$X_2(k) = \frac{1}{2j}\left[X((k-k_0))_N - X((k+k_0))_N\right]$$

Also, $$x(n)\cos\left(\frac{4\pi}{N}n\right) \xleftrightarrow[N]{DFT} \frac{1}{2}\left[X((k-2))_N + X((k+2))_N\right]$$

$$x(n)\sin\left(\frac{8\pi}{N}n\right) \xleftrightarrow[N]{DFT} \frac{1}{2j}\left[X((k-4))_N - X((k+4))_N\right]$$

6) CIRCULAR CONVOLUTION / TIME-DOMAIN CONVOLUTION OR FREQUENCY DOMAIN MULTIPLICATION

STATEMENT:

If $x(n) \xleftrightarrow{DFT} X(k)$

& $h(n) \xleftrightarrow{DFT} H(k)$

then $y(n) = x(n) \circledN h(n) \xleftrightarrow{DFT} X(k)H(k) = Y(k)$

Proof:

Consider two sequences $x(n)$ & $h(n)$ of length N, then circular convolution is defined as,

$$y(n) = x(n) \circledN h(n) = \sum_{m=0}^{N-1} x(m)\, h((n-m))_N$$

$$DFT\{y(n)\} = Y(k) = \sum_{n=0}^{N-1} y(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x(m)\, h((n-m))_N \cdot W_N^{kn}$$

Interchanging order of summations,

$$= \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} h((n-m))_N\, W_N^{kn}$$

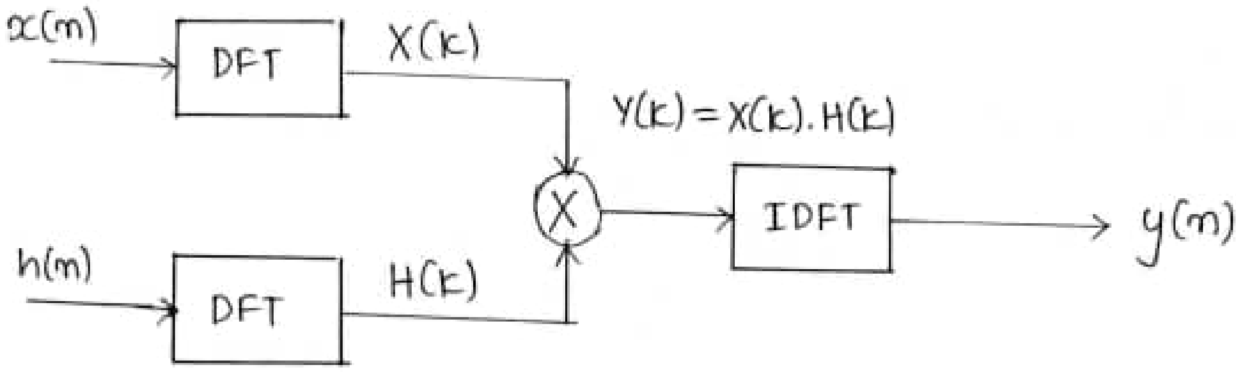
$$= \sum_{m=0}^{N-1} x(m)\, W_N^{km} H(k)$$

Using Time-shift property, $h((n-m))_N \xleftrightarrow{DFT} W_N^{km} H(k)$

$$= X(k)\, H(k)$$

Hence the proof.

II METHOD : Circular convolution using DFT & IDFT (stockham's method)



The method involves taking the N point DFTs of $x(n)$ & $h(n)$ both of length N points.

The respective DFTs are multiplied elementwise.

Then taking IDFT of the sequence $Y(k)$ to obtain $y(n)$.

(21) Compute the circular convolution of sequences $x(n) = \{1, 2, 3, 4\}$ and $h(n) = \{1, 2, 2\}$.

Soln:- Given $x(n) = \{1, 2, 3, 4\}$

& $h(n) = \{1, 2, 2\}$

Length of $x(n)$, $N_1 = 4$. Length of $h(n)$, $N_2 = 3$.

The convolution length $N = \max(N_1, N_2)$

$= \max(4, 3)$

$$\boxed{N = 4}$$

Since $h(n)$ is of length 3, pad one zero.

$\therefore h(n) = \{1, 2, 2, 0\}$

I- METHOD : Time-Domain Approach or Concentric circle Method

From the defn of circular convolution,

$$y(n) = x(n) \textcircled{N} h(n) = \sum_{m=0}^{N-1} x(m)\, h((n-m))_N$$

where $n = 0$ to $N-1$.

Given $N=4$,

$$\therefore y(n) = \sum_{m=0}^{3} x(m)\, h((n-m))_4, \quad n = 0,1,2,3$$

$$= x(0)\, h((n))_4 + x(1)\, h((n-1))_4 + x(2)\, h((n-2))_4 + x(3)\, h((n-3))_4$$

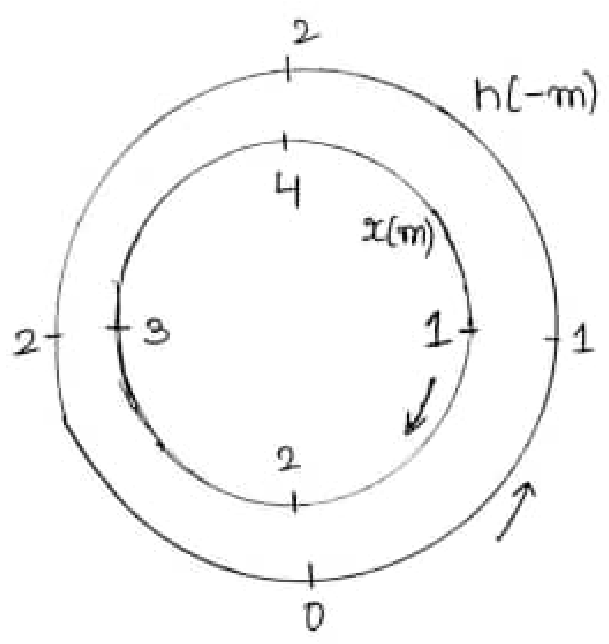
at $n=0$, $y(0) = x(0)h(0) + x(1)\, h((-1))_4 + x(2)\, h((-2))_4 + x(3)\, h((-3))_4$

$$y(0) = x(0)h(0) + x(1)h(3) + x(2)h(2) + x(3)h(1)$$

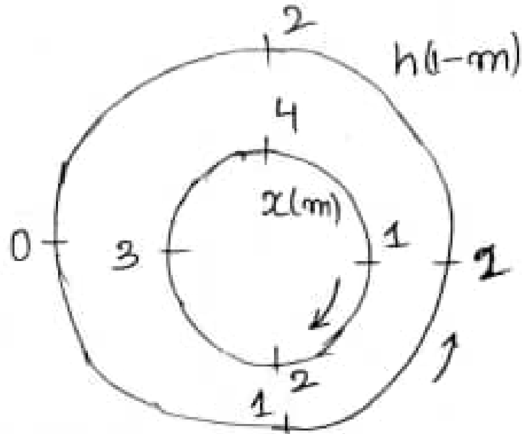
$$= (1)(1) + (2)(0) + (3)(2) + (4)(2)$$

$$= 1 + 0 + 6 + 8$$

$$y(0) = \underline{\underline{15}}$$



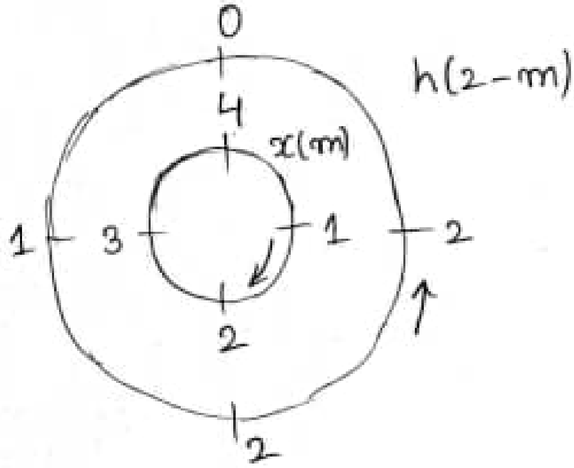
at $n=1$, $y(1) = x(0)\,h((1))_4 + x(1)\,h(0) + x(2)\,h(3) + x(3)\,h(2)$



$y(1) = 2 + 2 + 0 + 8$

$y(1) = \underline{\underline{12}}$

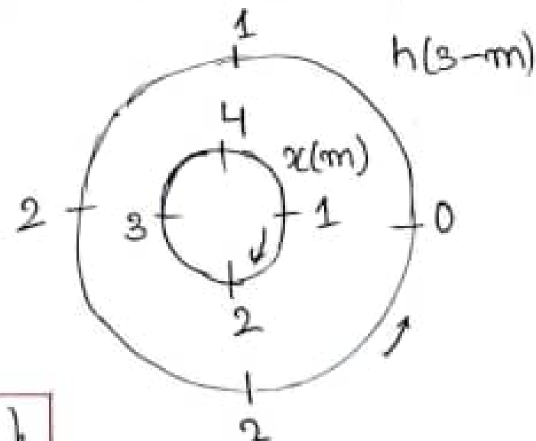
at $n=2$, $y(2) = x(0)\,h(2) + x(1)\,h(1) + x(2)\,h(0) + x(3)\,h(3)$



$y(2) = 2 + 4 + 3 + 0$

$y(2) = \underline{\underline{9}}$

at $n=3$, $y(3) = x(0)\,h(3) + x(1)\,h(2) + x(2)\,h(1) + x(3)\,h(0)$



$y(3) = 0 + 4 + 6 + 4$

$y(3) = \underline{\underline{14}}$

$\therefore \boxed{y(n) = \{15,\ 12,\ 9,\ 14\}}$

verification:

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \\ 9 \\ 14 \end{bmatrix}$$

II - METHOD : Using DFT & IDFT equations or Transform Domain Approach or Stockham's method

$$y(n) = x(n) \, \textcircled{N} \, h(n)$$

Applying DFT,

$$Y(k) = X(k) \cdot H(k)$$

(i) To find X(k). $X_N = x_N W_N$

$$X(k) = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\boxed{X(k) = \{10, -2+2j, -2, -2-2j\}}$$

(ii) To find H(k). $H_N = h_N W_N$

$$H(k) = \begin{bmatrix} 1 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\boxed{H(k) = \{5, -1-2j, 1, -1+2j\}}$$

(iii) Find the product of the sequence $X(k)$ & $H(k)$.

$$Y(k) = X(k) \cdot H(k)$$

$$\boxed{Y(k) = \{50,\ 6+2j,\ -2,\ 6-2j\}}$$

(iv) To find $y(n)$

$$y(n) = \text{IDFT}\{Y(k)\} = \frac{1}{N} Y_N W_N^*$$

$$y(n) = \frac{1}{4}\begin{bmatrix} 50 & 6+2j & -2 & 6-2j \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\therefore \boxed{y(n) = \{15,\ 12,\ 9,\ 14\}}$$

7) MULTIPLICATION OF TWO SEQUENCES OR MODULATION PROPERTY OR CONVOLUTION IN FREQUENCY DOMAIN

STATEMENT : If $x_1(n) \xleftrightarrow{DFT} X_1(k)$

& $x_2(n) \xleftrightarrow{DFT} X_2(k)$

then $x_1(n)\,x_2(n) \xleftrightarrow{DFT} \frac{1}{N}\left[X_1(k) \, \textcircled{N} \, X_2(k)\right]$

Proof:— Consider two sequences $x_1(n)$ & $x_2(n)$ of length N, with DFT $X_1(k)$ & $X_2(k)$.

$$\text{IDFT}\{X_1(k)\} = x_1(n) = \frac{1}{N}\sum_{l=0}^{N-1} X_1(l)\, W_N^{-ln}$$

$$\text{IDFT}\{X_2(k)\} = x_2(n) = \frac{1}{N}\sum_{m=0}^{N-1} X_2(m)\, W_N^{-mn}$$

Let $y(n) = x_1(n).\,x_2(n)$

then,

$$\text{DFT}\{y(n)\} = Y(k) = \sum_{n=0}^{N-1} y(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x_1(n)\, x_2(n)\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{l=0}^{N-1} X_1(l)\, W_N^{-ln} \right\} \cdot \left\{ \frac{1}{N} \sum_{m=0}^{N-1} X_2(m)\, W_N^{-mn} \right\} \times W_N^{kn}$$

$$= \frac{1}{N^2} \sum_{l=0}^{N-1} X_1(l) \cdot \sum_{m=0}^{N-1} X_2(m) \sum_{n=0}^{N-1} W_N^{(k-l-m)n}$$

If, $k-l-m = PN$

or $m = k-l-PN = ((k-l))_N$

or $l = k-m-PN = ((k-m))_N$

then, $\sum_{n=0}^{N-1} W_N^{(k-l-m)n} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N} PNn}$

$$= \sum_{n=0}^{N-1} 1 = N$$

$$\therefore\ Y(k) = \frac{1}{N^{2}} \sum_{l=0}^{N-1} X_1(l)\, X_2((k-l))_N \cdot N$$

$$\boxed{Y(k) = \frac{1}{N}\left[X_1(k) \circledN X_2(k) \right]}$$

8) SYMMETRY PROPERTY OF A COMPLEX VALUED SEQUENCE

STATEMENT: If $x(n) \xleftrightarrow{DFT} X(k)$

then $x^*(n) \xleftrightarrow{DFT} X^*(N-k) = X^*((-k))_N$

or $x^*((-n))_N = x^*(N-n) \xleftrightarrow{DFT} X^*(k)$

Proof: From defn of DFT, we know that

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Taking complex conjugate on both sides,

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n)\, W_N^{-kn}$$

replace k by N-k,

$$X^*(N-k) = \sum_{n=0}^{N-1} x^*(n)\, W_N^{-(N-k)n}$$

$$= \sum_{n=0}^{N-1} x^*(n)\, W_N^{-Nn} \cdot W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x^*(n)\, W_N^{kn} \qquad \because W_N^{-Nn} = 1$$

$$X^*(N-k) = DFT\{x^*(n)\}$$

or $x^*(n) \xleftrightarrow[N]{DFT} X^*(N-k)$

9) SYMMETRY PROPERTY OF REAL-VALUED SEQUENCE

STATEMENT: If $x(n) \xleftrightarrow{DFT} X(k)$

and $x(n)$ is real, then

$$X(k) = X^*(N-k)$$

Proof:- W.K.T, $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$

Taking complex conjugate on both sides,

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n)\, W_N^{-kn}$$

Since $x(n)$ is real, $x^*(n) = x(n)$

$$X^*(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{-kn} \cdot W_N^{Nn} \qquad \because W_N^{Nn} = 1$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{(N-k)n}$$

$$X^*(k) = X(N-k)$$

or $\boxed{X(k) = X^*(N-k)}$

(22) The first five points of the 8-point DFT of a real-valued sequence are

$\{ 0.25, (0.125 - j0.3018), 0, (0.125 - j0.0518), 0 \}$

Determining the remaining 3 points.

Soln:- Given,

$X(0) = 0.25$ $\qquad$ $X(3) = 0.125 - j0.518$

$X(1) = 0.125 - j0.3018$ $\qquad$ $X(4) = 0$

$X(2) = 0$

Since $x(n)$ is real valued

$$X^*(K) = X(N-K)$$

$$\text{or} \quad X(K) = X^*(N-K)$$

Here, $N=8$,

Then, for $K=5$, $X(5) = X^*(8-5) = X^*(3)$

$$= (0.125 - j0.0518)^*$$

$$X(5) = \underline{0.125 + j0.0518}$$

for $K=6$, $X(6) = X^*(8-6) = X^*(2)$

$$= \underline{0}$$

for $K=7$, $X(7) = X^*(8-7) = X^*(1)$

$$= (0.125 - j0.3018)^*$$

$$= \underline{0.125 + j0.3018}$$

10) CIRCULAR CORRELATION

For complex valued sequences $x(n)$ and $y(n)$

If $x(n) \xleftrightarrow{DFT} X(k)$

$y(n) \xleftrightarrow{DFT} Y(k)$

then $\tilde{r}_{xy}(l) \xleftrightarrow{DFT} \tilde{R}_{xy}(l) = X(k).Y^*(k)$

where $\tilde{r}_{xy}(l)$ is the circular cross-correlation sequence defined by

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

Proof: We can express $\tilde{r}_{xy}(l)$ as the circular convolution of $x(n)$ with $y^*(-n)$.

i.e, $\tilde{r}_{xy}(l) = x(l) \circledN y^*(-l)$

taking DFT on both sides,

$$\text{DFT}\{\tilde{r}_{xy}(l)\} = \text{DFT}\{x(l) \circledN y^*(-l)\}$$

$$\boxed{\tilde{R}_{xy}(k) = X(k).Y^*(k)}$$

using circular convolution & complex conjugate property

If $x(n) = y(n)$,

$$\tilde{r}_{xx}(l) \xleftrightarrow{DFT} \tilde{R}_{xx}(k) = |X(k)|^2$$

11) PARSEVAL'S THEOREM

For complex valued sequence $x(n)$ & $y(n)$

If $x(n) \xleftrightarrow{DFT} X(k)$

$y(n) \xleftrightarrow{DFT} Y(k)$

then $$\sum_{n=0}^{N-1} x(n).y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k).Y^*(k)$$

Proof:- Circular cross correlation of two sequences is defined as

$$\tilde{r_{xy}}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

at $l=0$, $$\tilde{r_{xy}}(0) = \sum_{n=0}^{N-1} x(n)\, y^*(n) \quad \text{—— (a)}$$

W.K.T, $$\text{IDFT}\{\tilde{R_{xy}}(k)\} = \tilde{r_{xy}}(l)$$

or $$\tilde{r_{xy}}(l) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{R_{xy}}(k)\, e^{j\frac{2\pi}{N}kl}$$ from defn of IDFT

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)\, e^{j\frac{2\pi}{N}kl}$$ from circular correlation property

at $l=0$,

$$\tilde{r_{xy}}(0) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k) \quad \text{—— (b)}$$

Equating RHS of Eqn's ⓐ & ⓑ,

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$$

If $x(n) = y(n)$, then

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

which is the energy in the finite duration sequence $x(n)$ in terms of the frequency components $X(k)$.

(23) Determine N-point circular correlation of $x(n) = \cos \frac{2\pi}{N} n$ & $y(n) = \sin \frac{2\pi}{N} n$.

Soln:- Given $x(n) = \cos \frac{2\pi n}{N}$

Taking N-point DFT,

$$X(k) = \frac{N}{2} \left[\delta(k-1) + \delta(k+1)\right]$$

Given $y(n) = \sin \frac{2\pi n}{N}$

Its N-point DFT, $Y(k) = \frac{N}{2j} \left[\delta(k-1) - \delta(k+1)\right]$

From circular correlation property,

$$R_{x\tilde{y}}(k) = X(k) \cdot Y^*(k)$$

$$R_{x\tilde{y}}(k) = \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right] \cdot \left\{\frac{N}{2j}\left[\delta(k-1) - \delta(k+1)\right]\right\}^*$$

$$= \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]\left\{\frac{-N}{2j}\left[\delta(k-1) - \delta(k+1)\right]\right\}$$

$$= -\frac{N^2}{4j}\left\{\left[\delta(k-1)\right]^2 + \delta(k+1)\delta(k-1) - \delta(k-1)\delta(k+1) - \left[\delta(k+1)\right]^2\right\}$$

WKT, $\delta(k+1)\,\delta(k-1) = 0$, $\left[\delta(k-1)\right]^2 = \delta(k-1)$

and $\left[\delta(k+1)\right]^2 = \delta(k+1)$.

$$= -\frac{N^2}{4j}\left[\delta(k-1) - \delta(k+1)\right]$$

$$R_{x\tilde{y}}(k) = \frac{-N}{2}\left\{\frac{N}{2j}\left[\delta(k-1) - \delta(k+1)\right]\right\}$$

taking IDFT,

$$r_{x\tilde{y}}(l) = -\frac{N}{2}\sin\frac{2\pi n}{N}$$

(24) Find the circular autocorrelation of $x(n) = \cos \frac{2\pi}{N} n$

Soln:— Given $x(n) = \cos\left(\frac{2\pi}{N} n\right)$

Taking DFT,

$$X(k) = \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]$$

WKT $\tilde{R}_{xy}(k) = X(k).Y^*(k)$

since $x(n) = y(n)$,

$$\tilde{R}_{xx}(k) = X(k).X^*(k)$$

$$\tilde{R}_{xx}(k) = \left\{\frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]\right\}\left\{\frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]\right\}^*$$

$$= \frac{N}{2} \cdot \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right]$$

taking IDFT,

$$\boxed{\tilde{r}_{xx}(n) = \frac{N}{2}\cos\left(\frac{2\pi}{N} n\right)}$$

(25) Find circular autocorrelation of the sequence

$x(n) = \{1, 2, 3, 4\}$.

Soln:- Given $x(n) = \{1, 2, 3, 4\}$, $N=4$

Find the 4-point DFT,

we get $X(k) = \{10, -2+2j, -2, -2-2j\}$

Autocorrelation of the sequence $x(n)$ is

$$\tilde{r}_{xx}(l) \xleftrightarrow{DFT} \tilde{R}_{xx}(k) = X(k) \cdot X^*(k)$$

$$= |X(k)|^2$$

$$\tilde{R}_{xx}(0) = |X(0)|^2 = 10^2 = 100$$

$$\tilde{R}_{xx}(1) = |X(1)|^2 = |(-2+2j)|^2$$

$$= |\sqrt{4+4}|^2 = 8$$

$$\tilde{R}_{xx}(2) = |X(2)|^2 = |-2|^2 = 4$$

$$\tilde{R}_{xx}(3) = |X(3)|^2 = |(-2-2j)|^2 = 8$$

$$\therefore \boxed{\tilde{R}_{xx}(k) = \{100, 8, 4, 8\}}$$

To find $r_{x\tilde{x}}(l)$:

$$r_{x\tilde{x}}(l) = \text{IDFT}\{R_{x\tilde{x}}(k)\}$$

$$r_{x\tilde{x}}(l) = \frac{1}{4}\begin{bmatrix}100 & 8 & 4 & 8\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & j & -1 & -j\\ 1 & -1 & 1 & -1\\ 1 & -j & -1 & j\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}120 & 96 & 88 & 96\end{bmatrix}$$

$$\boxed{r_{x\tilde{x}}(l) = \{30, 24, 22, 24\}}$$

(26) Find the circular correlation given $x(n) = \{1, 2, 3, 4\}$ and $y(n) = \{1, 2, 2, 0\}$

Soln:- Find $X(k)$

$$X(k) = \{10, -2+2j, -2, -2-2j\}$$

Find $Y(k)$

$$Y(k) = \{5, -1-2j, 1, -1+2j\}$$

$$\Rightarrow Y^*(k) = \{5, -1+2j, 1, -1-2j\}$$

Circular correlation property,

$$R_{x\tilde{y}}(k) = X(k) \cdot Y^*(k)$$

$$= \{ 50, -2-6j, -2, -2+6j \}$$

Taking IDFT,

$$\boxed{r_{x\tilde{y}}(l) = \{ 11, 16, 13, 10 \}}$$

(27) Given $x(n) = \{1, 2, 3, 4\}$ find the energy and hence verify Parseval's theorem.

Soln:- Given $x(n) = \{1, 2, 3, 4\}$

Energy of the signal $x(n)$ is given by

$$E = \sum_{n=-\infty}^{+\infty} |x(n)|^2$$

$$= \sum_{n=0}^{3} |x(n)|^2 = |x(0)|^2 + |x(1)|^2 + |x(2)|^2 + |x(3)|^2$$

$$= 1^2 + 2^2 + 3^2 + 4^2$$

$$\boxed{E = 30}$$

Find DFT of $x(n)$,

$$X(k) = \{10, -2+2j, -2, -2-2j\}$$

From Parseval's Theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$E = \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2$$

$$= \frac{1}{4} \left[|X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2 \right]$$

$$= \frac{1}{4} \left[100 + 8 + 4 + 8 \right]$$

$$E = \frac{120}{4} = \underline{\underline{30}}$$

Hence Parseval's Theorem is verified.

Definition of Even and Odd Symmetry and time reversal.

An N-point sequence is said to be circularly even if it is symmetric about the point zero on the circle.

ie, $\boxed{x(N-n) = x(n)}$, $1 \le n \le N-1$

An N-point sequence is said to be circularly odd if it is antisymmetric about the point zero on the circle.

ie, $\boxed{x(N-n) = -x(n)}$, $1 \le n \le N-1$

The time reversal of an N-point sequence is attained by reversing its samples about the point zero on the circle. Thus the sequence $x((-n))_N$ is given by

$$\boxed{x((-n))_N = x(N-n)}, \quad 0 \le n \le N-1$$

For periodic sequence $x_p(n)$, defn of even and odd sequences :

even : $x_p(n) = x_p(-n) = x_p(N-n)$

odd : $x_p(n) = -x_p(-n) = -x_p(N-n)$

If $x_p(n)$ is complex valued, then

conjugate even : $x_p(n) = x_p^*(N-n)$

conjugate odd : $x_p(n) = -x_p^*(N-n)$

Sequence $x_p(n)$ can be decomposed as

$$x_p(n) = x_{pe}(n) + x_{po}(n)$$

where

$$x_{pe}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(N-n)\right]$$

$$x_{po}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(N-n)\right]$$

Symmetry properties of the DFT

Let us assume that $x(n)$ and its DFT $X(k)$ are both complex valued.

Then, $x(n) = x_R(n) + j\, x_I(n)$, $n = 0, 1, \ldots N-1$ —— ①

$X(k) = X_R(k) + j\, X_I(k)$, $k = 0, 1, \ldots N-1$ —— ②

From the defn of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \text{——} ③$$

substituting eqn ① in eqn ③,

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n)\right] e^{-j\frac{2\pi}{N}kn}$$

and since, $e^{j\theta} = \cos\theta + j\sin\theta$

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + j\, x_I(n) \right] \left[\cos \frac{2\pi}{N} kn - j \sin \frac{2\pi}{N} kn \right]$$

$$= \sum_{n=0}^{N-1} \left[x_R(n) \cos\left(\frac{2\pi}{N} kn\right) - j\, x_R(n) \sin\left(\frac{2\pi}{N} kn\right) + j\, x_I(n) \cos\left(\frac{2\pi}{N} kn\right) + x_I(n) \sin\left(\frac{2\pi}{N} kn\right) \right] \quad \text{--- (4)}$$

Comparing eqn's (2) & (4),

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos\left(\frac{2\pi}{N} kn\right) + x_I(n) \sin\left(\frac{2\pi}{N} kn\right) \right] \quad \text{--- (5)}$$

$$X_I(k) = \sum_{n=0}^{N-1} \left[-x_R(n) \sin\left(\frac{2\pi}{N} kn\right) + x_I(n) \cos\left(\frac{2\pi}{N} kn\right) \right] \quad \text{--- (6)}$$

III^{ly}, WKT $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$

Substituting eqn (2) in above equation,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) + j\, X_I(k) \right] e^{j\frac{2\pi}{N} kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) + j\, X_I(k) \right] \left[\cos\left(\frac{2\pi}{N} kn\right) + j \sin\left(\frac{2\pi}{N} kn\right) \right]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \Big[X_R(k) \cos\left(\frac{2\pi}{N} km\right) + j X_R(k) \sin\left(\frac{2\pi}{N} km\right) + j X_I(k) \cos\left(\frac{2\pi}{N} km\right) - X_I(k) \sin\left(\frac{2\pi}{N} km\right) \Big] \quad \text{--- (7)}$$

Comparing eqn's (1) & (7),

$$x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos\left(\frac{2\pi}{N} kn\right) - X_I(k) \sin\left(\frac{2\pi}{N} kn\right) \right] \quad \text{--- (8)}$$

$$x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin\left(\frac{2\pi}{N} kn\right) + X_I(k) \cos\left(\frac{2\pi}{N} kn\right) \right] \quad \text{--- (9)}$$

case 1) <u>Real Valued sequences</u>

If $x(n)$ is real, then

$$X(N-k) = X^*(k) = X(-k)$$

And, $|X(N-k)| = |X(k)|$ also $\angle X(N-k) = -\angle X(k)$

For real $x(n)$, $x_I(n) = 0$, hence $x(n)$ can be determined from eqn (8).

Case 2) Real and even sequence

If $x(n)$ is real and even, that is

$$x(n) = x(N-n), \quad 0 \le n \le N-1$$

then eqn. ⑥ yields $X_I(k) = 0$. Hence the DFT reduces to

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right)}, \quad 0 \le k \le N-1$$

where $X(k)$ is real valued and even.

Since $X_I(k) = 0$, the IDFT reduces to

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos\left(\frac{2\pi kn}{N}\right)}, \quad 0 \le n \le N-1$$

Case 3) Real and odd sequence

If $x(n)$ is real and odd, that is

$$x(n) = -x(N-n), \quad 0 \le n \le N-1$$

then eqn ⑤ yields $X_R(k) = 0$. Hence

$$\boxed{X(k) = -j \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi kn}{N}\right)}, \quad 0 \le k \le N-1$$

which is purely imaginary and odd.

Since $X_R(k) = 0$, the IDFT reduces to

$$\boxed{x(n) = j\frac{1}{N} \sum_{k=0}^{N-1} X(k) \sin\left(\frac{2\pi kn}{N}\right)} \quad 0 \le n \le N-1$$

Case 4) <u>Purely Imaginary sequence</u>

In this case $x(n) = j\, x_I(n)$.

Eqn's (5) & (6) reduces to

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi k n}{N}$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi k n}{N}$$

We observe that $X_R(k)$ is odd and $X_I(k)$ is even.

If $x_I(n)$ is odd, then $X_I(k) = 0$, hence $X(k)$ is purely real.

If $x_I(n)$ is even, then $X_R(k) = 0$, hence $X(k)$ is purely imaginary.

The symmetry properties above may be summarised as follows:

$$x(n) = x_R^e(n) + x_R^o(n) + j\, x_I^e(n) + j\, x_I^o(n)$$

$$X(k) = X_R^e(k) + X_R^o(k) + j\, X_I^e(k) + j\, X_I^o(k)$$

PROPERTIES OF THE DFT

PROPERTY	TIME DOMAIN	FREQUENCY DOMAIN
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(k) + a_2 X_2(k)$
Time Reversal	$x(N-n)$	$X(N-k)$
Circular Time Shift	$x((n-l))_N$	$X(k).e^{-j\frac{2\pi}{N}kl}$
Circular Frequency shift	$x(n)\, e^{j\frac{2\pi}{N}ln}$	$X((k-l))_N$
Complex Conjugate	$x^*(n)$	$X^*(N-k)$
Circular Convolution	$x_1(n) \textcircled{N} x_2(n)$	$X_1(k)\, X_2(k)$
Circular Correlation	$x(n) \textcircled{N} y^*(-n)$	$X(k)\, Y^*(k)$
Multiplication of two sequences	$x_1(n)\, x_2(n)$	$\frac{1}{N}\left[X_1(k) \textcircled{N} X_2(k)\right]$
Parseval's Theorem	$\sum_{n=0}^{N-1} x(n)\, y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k)$

(28) A 498 point DFT of a real valued sequence $x(n)$ has the following DFT samples

$X(0) = 2$ $\quad$ $X(249) = 2.9$

$X(11) = 7 + j3.1$ $\quad$ $X(309) = -4.7 - j1.9$

$X(k_1) = -2.2 - j1.5$ $\quad$ $X(k_3) = 3 - j0.7$

$X(112) = 3 + j0.7$ $\quad$ $X(412) = -2.2 + j1.5$

$X(k_2) = -4.7 + j1.9$ $\quad$ $X(k_4) = 7 - j3.1$

The other samples have a value zero. Find the value of k_1, k_2, k_3 & k_4.

Sol'n :- For real valued sequence,

WKT, $\quad X(k) = X^*(N-k)$

Given $N = 498$.

a) For $k = 412$, $\quad X(412) = X^*(498 - 412)$

$$= X^*(86)$$

or $X(86) = X^*(412)$

$$= (-2.2 + j1.5)^*$$

$$X(86) = -2.2 - j1.5 = X(k_1)$$

$\therefore$ $\boxed{k_1 = 86}$

b) for $k = 309$, $X(309) = X^*(498 - 309)$

$= X^*(189)$

or $X(189) = X^*(309)$

$= (-4.7 - j1.9)^*$

$X(189) = -4.7 + j1.9 = X(k_2)$

$\therefore \boxed{k_2 = 189}$

c) for $k = 112$, $X(112) = X^*(498 - 112)$

$= X^*(386)$

or $X(386) = X^*(112)$

$= (3 + j0.7)^*$

$X(386) = 3 - j0.7 = X(k_3)$

$\therefore \boxed{k_3 = 386}$

d) for $k = 11$, $X(11) = X^*(498 - 11)$

$= X^*(487)$

or $X(487) = X^*(11)$

$= (7 + j3.1)^*$

$X(487) = 7 - j3.1 = X(k_4)$

$\therefore \boxed{k_4 = 487}$

Hence, $k_1 = 86$, $k_2 = 386$, $k_3 = 189$ & $k_4 = 487$

(29) Find the 4-point DFTs of the two sequences $x(n)$ & $y(n)$ using a single 4 point DFT.

$x(n) = \{1, 2, 0, 1\}$ & $y(n) = \{2, 2, 1, 1\}$.

Sol'n:-

From the given, combining the two sequences $x(n)$ and $y(n)$ to create a complex sequence $h(n)$.

With, $h(n) = x(n) + j\,y(n)$, $0 \le n \le 3$

$\therefore h(n) = \{(1+2j), (2+2j), j, (1+j)\}$

Taking DFT, $H_N = h_N W_N$

$$\therefore H(K) = \begin{bmatrix} 1+2j & 2+2j & j & 1+j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$H(k) = \begin{bmatrix} 4+6j & 2 & -2 & 2j \end{bmatrix}$$

$$\Rightarrow H^*(k) = \begin{bmatrix} 4-6j & 2 & -2 & -2j \end{bmatrix}$$

Using the relation, $X(k) = \dfrac{H(k) + H^*((-k))_N}{2}$

for $k=0$, $X(0) = \dfrac{H(0) + H^*(0)}{2} = \dfrac{(4+6j)+(4-6j)}{2}$

$= \dfrac{8}{2} = \underline{\underline{4}}$

for $k=1$, $X(1) = \dfrac{H(1) + H^*((-1))_4}{2} = \dfrac{H(1) + H^*(3)}{2}$

$= \dfrac{2-2j}{2} = \underline{\underline{1-j}}$

for $k=2$, $X(2) = \frac{H(2) + H^*((-2))_4}{2} = \frac{H(2) + H^*(2)}{2}$

$= \frac{-2-2}{2} = \underline{\underline{-2}}$

for $k=3$, $X(3) = \frac{H(3) + H^*((-3))_4}{2} = \frac{H(3) + H^*(1)}{2}$

$= \frac{2j+2}{2} = \underline{\underline{1+j}}$

$\therefore \boxed{X(k) = \{4, (1-j), -2, (1+j)\}}$

Also, $Y(k) = \frac{H(k) - H^*((-k))_N}{2j}$

for $k=0$, $Y(0) = \frac{H(0) - H^*(0)}{2j} = \frac{4+6j-4+6j}{2j} = \underline{\underline{6}}$

for $k=1$, $Y(1) = \frac{H(1) - H^*((-1))_4}{2j} = \frac{H(1) - H^*(3)}{2j}$

$= \frac{2+2j}{2j} = \underline{\underline{1-j}}$

for $k=2$, $Y(2) = \frac{H(2) - H^*((-2))_4}{2j} = \frac{H(2) - H^*(2)}{2j}$

$= \frac{-2+2}{2j} = \underline{\underline{0}}$

for $k=3$, $Y(3) = \frac{H(3) - H^*((-3))_4}{2j} = \frac{H(3) - H^*(1)}{2j}$

$= \frac{2j-2}{2j} = \underline{\underline{1+j}}$

$\therefore \boxed{Y(k) = \{6, (1-j), 0, (1+j)\}}$

(30) Let $x_p(n)$ be a periodic sequence with fundamental period N. If the N-point DFT $\{x_p(n)\} = X_1(k)$ and 3N-point DFT $\{x_p(n)\} = X_3(k)$.

(i) Find the relationship between $X_1(k)$ & $X_3(k)$.

(ii) Verify the above result for $\{2,1\}$ & $\{2,1,2,1,2,1\}$.

<u>soln:-</u> (i) WKT N-point DFT $\{x(n)\} = X(k)$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$\text{N-point DFT}\{x_p(n)\} = X_1(k) = \sum_{n=0}^{N-1} x_p(n)\, W_N^{kn}$$

$$\text{3N-point DFT}\{x_p(n)\} = X_3(k) = \sum_{n=0}^{3N-1} x_p(n)\, W_{3N}^{kn}$$

$$X_3(k) = \sum_{n=0}^{N-1} x_p(n)\, W_{3N}^{kn} + \sum_{n=0}^{N-1} x_p(n+N)\, W_{3N}^{k(n+N)} + \sum_{n=0}^{N-1} x_p(n+2N)\, W_{3N}^{k(n+2N)}$$

$$= \sum_{n=0}^{N-1} x_p(n)\, W_{3N}^{kn} + \sum_{n=0}^{N-1} x_p(n)\, W_{3N}^{kn} \cdot W_{3N}^{kN} + \sum_{n=0}^{N-1} x_p(n)\, W_{3N}^{kn} \cdot W_{3N}^{2kN}$$

$\because$ From periodicity of input.

$x_p(n+N) = x_p(n)$

& $x_p(n+2N) = x_p(n)$

Also, $W_{3N}^{Nk} = e^{-j\frac{2\pi}{3N}Nk} = e^{-j\frac{2\pi}{3}k} = W_3^k$

|||ly $W_{3N}^{2Nk} = W_3^{2k}$ & $W_{3N}^{nk} = W_N^{n\frac{k}{3}}$.

$$\therefore X_3(k) = \sum_{n=0}^{N-1} x_p(n)\left[1 + W_3^k + W_3^{2k}\right] W_N^{n\frac{k}{3}}$$

$$= \left[1 + W_3^k + W_3^{2k}\right] \sum_{n=0}^{N-1} x_p(n)\, W_N^{n\frac{k}{3}}$$

$$\boxed{X_3(k) = \left[1 + W_3^k + W_3^{2k}\right] X_1(k/3)}$$

ii) Given $x_1(n) = \{2, 1\}$

Its DFT, $X_1(k) = 2 + W_2^k$

$x_3(n) = \{2, 1, 2, 1, 2, 1\}$ with DFT

$$X_3(k) = 2 + W_6^k + 2W_6^{2k} + W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$

$$W_6^k = W_{2\times 3}^k = W_2^{k/3}$$

Then, $X_3(k) = \left[2 + W_2^{k/3}\right] + W_6^{2k}\left[2 + W_2^{k/3}\right]$

$$+ W_6^{4k}\left[2 + W_2^{k/3}\right]$$

$$= \left[2 + W_2^{k/3}\right]\left[1 + W_6^{2k} + W_6^{4k}\right]$$

$$X_3(k) = X_1(k/3)\left[1 + W_3^k + W_3^{2k}\right]$$

Hence verified.

(31) Suppose that you are given a program to find the DFT of a complex-valued sequence $x(n)$. How can we use this program to find the inverse DFT of $X(k)$.

Sol'n:- WKT,

$$DFT\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \text{—} (1)$$

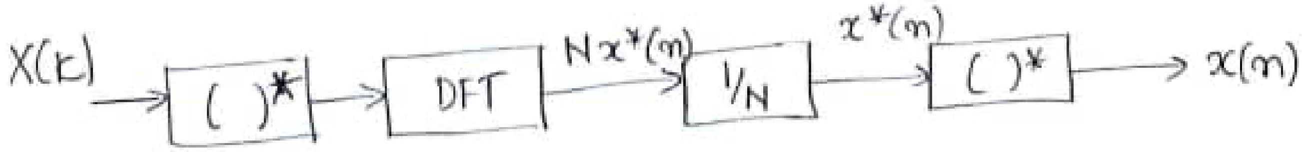
$$IDFT\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad \text{—} (2)$$

Taking complex conjugate of eq'n (2),

$$x^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) W_N^{kn}$$

or $$N x^*(n) = \sum_{k=0}^{N-1} X^*(k) W_N^{kn}$$

From observing the above eq'n, we can use DFT program to compute IDFT by calculating $X^*(k)$ and finding its DFT. The output is $N.x^*(n)$. Scale it by $\frac{1}{N}$ and take its conjugate.



(32) Consider the sequence $x(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$.

Let $X(k)$ be the 6-point DFT of $x(n)$. Find the finite length sequence $y(n)$ that has a 6-point DFT $Y(k) = W_6^{4k} X(k)$.

Soln:- Given, $x(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$

Its DFT,
$$X(k) = 4 + 3W_6^{k} + 2W_6^{2k} + W_6^{3k}$$

Also, given $Y(k) = W_6^{4k} X(k)$

$$Y(k) = W_6^{4k}\left[4 + 3W_6^{k} + 2W_6^{2k} + W_6^{3k}\right]$$

$$= 4W_6^{4k} + 3W_6^{5k} + 2W_6^{6k} + W_6^{7k}$$

WKT, $W_6^{6k} = W_6^{0} = 1$, $W_6^{7k} = W_6^{k}$

$$\therefore Y(k) = 4W_6^{4k} + 3W_6^{5k} + 2 + W_6^{k}$$

$$Y(k) = 2 + W_6^{k} + 4W_6^{4k} + 3W_6^{5k}$$

Taking IDFT,

$$y(n) = 2\delta(n) + \delta(n-1) + 4\delta(n-4) + 3\delta(n-5)$$

$$\therefore \boxed{y(n) = \{\underset{\uparrow}{2}, 1, 0, 0, 4, 3\}}$$

(33) If $x(n) = \{1, 2, 0, 3, -2, 4, 7, 5\}$.

Evaluate the following (i) $X(0)$ (ii) $X(4)$ (iii) $\sum_{k=0}^{7} X(k)$

(iv) $\sum_{k=0}^{7} |X(k)|^2$

Sol'n:- (i) From the def'n of DFT we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$N=8$,

at $k=0$, $$X(0) = \sum_{n=0}^{7} x(n)\, W_8^0$$

$$= \sum_{n=0}^{7} x(n) = 1+2+0+3-2+4+7+5$$

$$\boxed{X(0) = 20}$$

(ii) $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

for $N=8$, Let $k=4$

$$X(4) = \sum_{n=0}^{7} x(n)\, W_8^{4n}$$

$\because W_8^{4n} = e^{\frac{-j2\pi}{8}\times 4n}$

$= (e^{-j\pi})^n = (-1)^n$

$$= \sum_{n=0}^{7} x(n)\,(-1)^n$$

$$X(4) = x(0) - x(1) + x(2) - x(3) + x(4) - x(5) + x(6) - x(7)$$

$$\boxed{X(4) = -8}$$

iii) To find $\sum_{k=0}^{7} X(k)$

Consider the IDFT equation,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

for N=8, put n=0,

$$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k)$$

$$\therefore \sum_{k=0}^{7} X(k) = 8\, x(0) \quad = \underline{\underline{8}}$$

iv) To find $\sum_{k=0}^{7} |X(k)|^2$

From Parseval's Theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

or $$\sum_{k=0}^{N-1} |X(k)|^2 = N \sum_{n=0}^{N-1} |x(n)|^2$$

for N=8, $$\sum_{k=0}^{7} |X(k)|^2 = 8\left[\sum_{n=0}^{7} |x(n)|^2\right]$$

$$= 8\left[|1|^2 + |2|^2 + |0|^2 + |3|^2 + |-2|^2 + |4|^2 + |7|^2 + |5|^2\right]$$

$$= 8\left[1 + 4 + 9 + 4 + 16 + 49 + 25\right]$$

$$\sum_{k=0}^{7} |X(k)|^2 = \underline{\underline{864}}$$

(34) Let $x(n)$ be a finite length sequence with $X(k) = \{0, (1+j), 1, (1-j)\}$ using the properties of DFT, find DFTs of the following sequences.

(i) $x_1(n) = e^{j\frac{\pi}{2}n}\, x(n)$

(ii) $x_2(n) = \cos\frac{\pi}{2}n\; x(n)$

(iii) $x_3(n) = x((n-1))_4$

Sol'n:- (i) Given $x_1(n) = e^{j\frac{\pi}{2}n}\, x(n)$

$$\text{or } x_1(n) = e^{j\frac{2\pi}{4}n}\, x(n)$$

Applying DFT and using circular frequency shift property

$$X_1(k) = X((k-1))_4$$

$$\therefore X_1(k) = \{(1-j), 0, (1+j), 1\}$$

(ii) Given $x_2(n) = \cos\frac{\pi}{2}n\; x(n)$

$$\text{or } x_2(n) = \frac{1}{2}\left[e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right] x(n)$$

$$= \frac{1}{2}\left[e^{j\frac{2\pi}{4}n} + e^{-j\frac{2\pi}{4}n}\right] x(n)$$

Applying DFT on both sides and using circular frequency shift property.

$$X_2(k) = \frac{1}{2}\left[X((k-1))_4 + X((k+1))_4\right]$$

Given $X(k) = \{0, (1+j), 1, (1-j)\}$

$X((k-1))_4 = \{(1-j), 0, (1+j), 1\}$

$X((k+1))_4 = \{(1+j), 1, (1-j), 0\}$

$\therefore X_2(k) = \frac{1}{2}[2 \quad 1 \quad 2 \quad 1]$

$X_2(k) = \{1, 0.5, 1, 0.5\}$

iii) Given $x_3(n) = x((n-1))_4$

taking DFT on both sides and using circular time-shift property.

$$X_3(k) = W_4^k X(k)$$

at $k=0$, $X_3(0) = W_4^0 X(0) = 0$

at $k=1$, $X_3(1) = W_4^1 X(1) = (-j)(1+j) = (1-j)$

at $k=2$, $X_3(2) = W_4^2 X(2) = (-1)(1) = -1$

at $k=3$, $X_3(3) = W_4^3 X(3) = j(1-j) = 1+j$

$$\therefore X_3(k) = \{0, 1-j, -1, 1+j\}$$

(35) Let $x(n) = \{1, 2, 3, 4\}$ with $X(k) = \{10, -2+2j, -2, -2-2j\}$. Find the DFT of $x_1(n) = \{1, 0, 2, 0, 3, 0, 4, 0\}$ without actually calculating the DFT.

Soln:- $\text{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{N-1} x_1(n) W_N^{kn}$

Dividing above summation into even and odd parts.

$$X_1(k) = \sum_{n=0}^{3} x_1(2n) W_N^{2nk} + \sum_{n=0}^{3} x_1(2n+1) W_N^{(2n+1)k}$$

$N=8$,

$$X_1(k) = \sum_{n=0}^{3} x_1(2n) W_8^{2nk} + 0$$ $\because$ odd numbered samples are zero.

$$X_1(k) = \sum_{n=0}^{3} x(n) W_{8/2}^{nk}$$

$$X_1(k) = \sum_{n=0}^{3} x(n) W_4^{nk}$$ or $X_1(k) = \begin{cases} X(k), & n=0,1,2,3 \\ X(k), & n=4,5,6,7 \end{cases}$

$$X_1(k) = X(k)$$

ie, $X_1(k) = \{10, -2+2j, -2, -2-2j, 10, -2+2j, -2, -2-2j\}$

Using periodicity property,

$$X_1(4) = X(4) = X(0) \ldots\ldots$$

(36) Let $X(k)$ be a 14-point DFT of a length-14 real sequence $x(n)$. The first 8 samples of $X(k)$ are given by

$X(0) = 12$, $X(1) = -1+j3$, $X(2) = 3+j4$

$X(3) = 1-j5$, $X(4) = -2+j2$, $X(5) = 6+j3$

$X(6) = -2-j3$, $X(7) = 10$

Determine the remaining samples of $X(k)$.

Evaluate the following functions of $x(n)$ without computing the IDFT of $X(k)$.

(i) $x(0)$ (ii) $x(7)$ (iii) $\sum_{n=0}^{13} x(n)$

(iv) $\sum_{n=0}^{13} e^{\frac{j4\pi n}{7}} x(n)$ (v) $\sum_{n=0}^{13} |x(n)|^2$

Soln:- For a real valued sequence $x(n)$, symmetry property of real valued sequence.

$$X(k) = X^*(N-k) = X^*((-k))_N$$

$$X(8) = X^*(14-8) = X^*(6) = -2+j3$$

$$X(9) = X^*(14-9) = X^*(5) = 6-j3$$

$$X(10) = X^*(14-10) = X^*(4) = -2-j2$$

$$X(11) = X^*(14-11) = X^*(3) = 1+j5$$

$$X(12) = X^*(14-12) = X^*(2) = 3-j4$$

$$X(13) = X^*(14-13) = X^*(1) = -1-j3$$

(i) We know $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$; $0 \le n \le N-1$

$N = 14$, $x(n) = \frac{1}{14} \sum_{k=0}^{13} X(k) e^{j\frac{\pi}{7}kn}$

Put $n=0$, $x(0) = \frac{1}{14} \sum_{k=0}^{13} X(k)$

$= \frac{1}{14} \left[X(0) + X(1) + X(2) + \dots + X(13) \right]$

$x(0) = \frac{32}{14} = \underline{\underline{2.2857}}$

(ii) We know, $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$

$N=14$, $x(n) = \frac{1}{14} \sum_{k=0}^{13} X(k) e^{j\frac{2\pi}{14}kn}$

Put $n = 7$, $x(7) = \frac{1}{14} \sum_{k=0}^{13} (-1)^k X(k)$

$= \frac{-12}{14} = \underline{\underline{-0.8571}}$

(iii) $X(k) = \sum_{n=0}^{13} x(n) e^{-j\frac{2\pi}{14}kn}$

Put $k=0$, $X(0) = \sum_{n=0}^{13} x(n)$

$\therefore \sum_{n=0}^{13} x(n) = X(0) = \underline{\underline{12}}$

iv) The DFT of $e^{j\frac{4\pi}{7}n}\,x(n)$ or $e^{j\frac{2\pi}{14}4n}\,x(n)$ is

$$X((k-4))_{14}$$

We know, $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$; $0 \le k \le N-1$

$$\therefore X((k-4))_{14} = \sum_{n=0}^{13} \left\{ e^{j\frac{4\pi}{7}n}\, x(n) \right\} e^{-j\frac{2\pi}{14}kn}$$

Put $k=0$,

$$X((-4))_{14} = \sum_{n=0}^{13} e^{j\frac{4\pi}{7}n}\, x(n)$$

$$\therefore \sum_{n=0}^{13} e^{j\frac{4\pi}{7}n}\, x(n) = X(10)$$

$$= \underline{\underline{-2-j2}}$$

v) From Parseval's Theorem,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$N=14$,

$$\sum_{n=0}^{13} |x(n)|^2 = \frac{1}{14} \sum_{k=0}^{13} |X(k)|^2$$

$$= \frac{1}{14}\left[|X(0)|^2 + |X(1)|^2 + \dots \dots + |X(13)|^2 \right]$$

$$= \frac{498}{14} = \underline{\underline{35.5714}}$$

Model Question Paper with effect from 2023-24(CBCS Scheme)

USN | | | | | | | | | | |

Fifth Semester B.E. Degree Examination
Digital Signal Processing

Time: 03 Hours **Max. Marks: 100**

Note: Answer any FIVE full questions, choosing at least ONE question from each MODULE.

Sl No		Questions	BTL	Marks
		Module 1		
Q1	a	Determine the energy and power of the unit step sequence.	L2	4
	b	Consider an LTI system with input x(n) & unit impulse response h(n) given below, Compute y(n), $x(n) = 2^n u(-n)$ & $h(n) = u(n)$.	L3	8
	c	Define signal with example. Explain Classification of signals with examples also define Elementary Discrete-Time Signals.	L2	8
		OR		
Q2	a	Determine the response of the following systems to the input signal $x(n) = \begin{cases} \lvert n \rvert & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$ a) y(n) = x(n) b) y(n) = x(n-1) c) y(n) = x(n+1) d) y(n) = 1/3 [x(n+1) + x(n)+x(n-1)]	L2	4
	b	The impulse response of a linear time-invariant system is $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$. Determine the response of the system to the input signal x (n) = {1, 2, 3, 1}.	L3	8
	c	Define system with example. Explain Classification of Discrete-Time system with examples.	L2	8
		Module 2		
Q3	a	Determine the z-transform of the signal $x(n) = \alpha^n u(n) = \begin{cases} n, & n \ge 0 \\ 0, & n < 0 \end{cases}$ and $x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n > 0 \\ -\alpha^n, & n \le -1 \end{cases}$	L3	8
	b	Determine the z-transforms of the following finite-duration signals. $x_1(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$ (b) $x_2(n) = \{1, 2, 5, \underset{\uparrow}{7}, 0, 1\}$.	L3	4
	c	Mention the properties of Z transform with equations.	L2	8
		OR		
Q4	a	Perform Circular convolution of the following sequences using concentric circle method: $x_1(n) = \{2, 1, 2, 1\}$, $x_2(n) = \{1, 2, 3, 4\}$.	L3	8
	b	Find the DFT of the sequence $x(n) = \delta(n) + 2\delta(n-2) + \delta(n-3)$.	L3	4
	c	Explain Frequency Domain Sampling and Reconstruction of Discrete Time Signals with the help of necessary equations.	L2	8
		Module 3		
Q5	a	State and prove the following properties: i) Circular Time shift Property ii) Circular Frequency shift Property iii) Parsevals theorem iv) Complex conjugate property	L2	10
	b	Use the 8 point radix-2 DIT-FFT algorithm to find the DFT of the sequence x(n)={1,1,1,1,0,0,0,0}.	L3	10
		OR		

Q6	a	Illustrate the Inverse Decimation in Time FFT algorithm with the help of necessary equations and signal flow representation	L2	10
	b	Using linear convolution find y(n)= x(n) h(n) for the sequences x(n)={1, 2, -1, 2, 3, -2, -3, -1, 1, 1, 2, -1} & h(n)={1, 2}. Compare the result by solving the problem using overlap save & overlap add method.	L3	10
		Module 4		
Q7	a	The desired frequency response of a low pass filter is given by $H_d(e^{j\omega}) = H_d(\omega) = \begin{cases} e^{-j3\omega}, & \lvert\omega\rvert < 3\pi/4 \\ 0, & 3\pi/4 < \lvert\omega\rvert < \pi \end{cases}$ Determine the frequency response of the FIR filter if Hamming window is used with M=7.	L3	10
	b	Mention different windows with equations used in design of FIR filters.	L2	5
	c	Realize the system function $H(z) = 1 + 3/2\, z^{-1} + 4/5\, z^{-2} + 5/9\, z^{-3} + 1/9\, z^{-4}$ using direct form .	L2	5
		OR		
Q8	a	A filter is to be designed with the desired frequency response $H_d(e^{j\omega}) = H_d(\omega) = \begin{cases} 0, & -\pi/4 < \omega < \pi/4 \\ e^{-j2\omega}, & \pi/4 < \lvert\omega\rvert < \pi \end{cases}$ Find the frequency response of the FIR filter designed using a rectangular window defined below: $w_R(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$	L3	10
	b	Mention the Design steps followed in design of Linear Phase FIR Filter.	L2	5
	c	Realize a cascade form FIR filter for the following system function. $H(z) = (1 + \tfrac{1}{4} z^{-1} + z^{-2})\ (1 + \tfrac{1}{8} z^{-1} + z^{-2})$.	L2	5
		Module 5		
Q9	a	Design a digital lowpass Butterworth filter with the following specifications: 1. 3 dB attenuation at the passband frequency of 1.5 kHz 2. 10 dB stopband attenuation at the frequency of 3 kHz 3. Sampling frequency of 8,000 Hz.	L3	8
	b	The normalized low pass filter with a cutoff frequency of 1 rad/sec is given as: $H_P(s) = 1/(s+1)$ Use the given $H_p(s)$ and the BLT to design a corresponding digital IIR lowpass filter with a cutoff frequency of 15 Hz and a sampling rate of 90 Hz.	L3	7
	c	Explain Bilinear Transformation design procedure in designing IIR filters.	L2	5
		OR		
Q10	a	Obtain analog lowpass prototype transformation to the low pass, high pass, band pass filter, band stop filters.	L2	8
	b	Obtain direct form I and direct form II for the system described by $y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$.	L3	7
	c	Given the following IIR filter: $y(n) = 0.2x(n) + 0.4x(n-1) + 0.5y(n-1)$, Determine the transfer function, nonzero coefficients, and impulse response.	L2	5

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