AKSHAYA INSTITUTE OF TECHNOLOGY, TUMAKURU
Approved by AICTE, New Delhi, affiliated to VTU, Belgaum, Recognized by Government of Karnataka
Obalapura Post, Lingapura, Koratagere Road, Tumkur- 572 106, Karnataka

DEPARTMENT OF MATHEMATICS

SUBJECT: MATHEMATICS-I

MODULE: VECTOR SPACE & LINEARTRANSFORMATIONS

Prepared by:

DEPARTMENT OF MATHEMATICS

AIT

www.ait-tumkur.ac.in

DEPARTMENT OF MATHEMATICS

VISION AND MISSION OF THE DEPARTMENT

VISION OF THE DEPARTMENT:

The Mathematics Department offers a broad and challenging academic program that supports the vision of Visvesvaraya Technological University. We aim to provide high-quality education in undergraduate and graduate mathematics. Department of Mathematics strives to be recognized for academic excellence through the depth of its teaching and research.

MISSION OF THE DEPARTMENT:

- Making Engineers to develop mathematical thinking and applying it to solve complex engineering problems, designing mathematical modelling for systems involving global level technology.
- Create an environment of mutual respect, encouragement and openness in our classroom, respecting intellectual diversity and provide academic freedom to faculty in pursuit of original research ideas in their chosen areas of interest.

Module 3: Vector space and Linear Transformations:

Introduction :—

Group :- A non empty set equipped with one binary operation $*$ is called group if $*$ satisfies following postulates (or axioms).

1. Closure Property : $\forall$ a and $b \in G \Rightarrow a * b \in G$.
2. Associative Property : $\forall a, b, c \in G \Rightarrow a*(b*c) = (a*b)*c$
3. Existence of identity :
   There exists an element $e \in G$ such that
   $$a*e = a = e*a.$$
   The element e is called the identity.
4. Existence of inverse :
   Each element of G possesses inverse, i.e $\forall a \in G \Rightarrow$ there exists an element $b \in G$ such that
   $$a*b = e = b*a$$
   Then b is called inverse of a and we write $b = a^{-1}$
   $$\therefore a * a^{-1} = e = a^{-1} * a.$$

$*$ A group with commutative property is known as abelian group or commutative group.
i.e $\forall a, b \in G$, if $a*b = b*a$.
$\therefore$ '$*$' is commutative on G.

Eg:- i) $(I, +)$ is an abelian group as $(I,+)$ is a group and also $\forall a, b \in I$.

ii) $(Q_0, \cdot)$, $(R_0, \cdot)$, $(C_0, \cdot)$ are examples of abelian group.

Rings:-

Suppose R is a non-empty set equipped with two binary operation called addition and multiplication and denoted by '+' and '$\cdot$' respectively.

Then the algebraic structure $(R, +, \cdot)$ is known as ring if the following axioms are satisfied.

1) $(R, +)$ is an abelian group.

   i) closure property for '+'.

   $\forall\ a, b \in R \Rightarrow a+b \in R$.

   ii) Associative property for '+'

   $\forall\ a, b, c \in R \Rightarrow a+(b+c) = (a+b)+c$

   iii) Existence of identity element for '+'

   $\forall\ a \in R\ \exists\ 0 \in R \ni: a+0 = a = 0+a$.

   0 is known as additive element or zero element of the ring.

   iv) Existence of inverse element for '+'.

   $\forall\ a \in R\ \exists\ -a \in R$.

   such that $a+(-a) = 0 = -a+a$.

   $-a$ is known as additive inverse of a.

2) $(R, \cdot)$ is a semi group.

   i) closure property for '$\cdot$'

   $\forall\ a, b \in R \Rightarrow a \cdot b \in R$.

   ii) Associative property for '$\cdot$'

   $\forall\ a, b, c \in R \Rightarrow a\cdot(b\cdot c) = (a\cdot b)\cdot c$

3) Multiplication '$\cdot$' distributes over addition '+' from left and also from right.

   $\forall\ a, b, c \in R$ i) $a\cdot(b+c) = a\cdot b + a\cdot c$

   ii) $(a+b)\cdot c = a\cdot c + b\cdot c$

Eg:- $(I, +, \cdot)$ is a ring.

* **Commutative Ring:-**

Ring $(R, +, \cdot)$ is a commutative ring, if $\forall\, a, b \in R$, $a \cdot b = b \cdot a$.

Eg:- $(I, +, \cdot)$.

* **Ring With Unity :-**

Ring $(R, +, \cdot)$ is known as ring with unity if $\forall\, a \in R\ \exists\ 1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$.

* **Commutative Ring with Unity :-**

Ring $(R, +, \cdot)$ is commutative ring with unity, if $\forall\, a, b \in R \Rightarrow a \cdot b = b \cdot a$ and $\forall\, a \in R\ \exists\ 1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$.

Field :-

A commutative ring with unity in which every non-zero element possesses their multiplicative inverse is known as field.

i.e $(F, +, \cdot)$ is a field if

(i) $(F, +)$ is an abelian group.

(ii) $(F_0, \cdot)$ is an abelian group.

(iii) Multiplication distributes over addition from the left and also from the right.

Eg:- $(Q, +, \cdot)$, $(R, +, \cdot)$, $(C, +, \cdot)$ are examples of fields.

* **Note :-**

If $(F, +, \cdot)$ is a field then F contains at least two elements, zero element and unit element i.e additive identity element and multiplicative identity element.

Vector Spaces:-

Let F be a field. A vector space over F, is a non-empty set V together with two operations (called addition and scalar multiplication) such that for each $u, v \in V$ there is a unique element $u+v \in V$ and for each $\alpha \in F$ and $u \in V$ there is a unique element $\alpha u \in V$, and it satisfies the following conditions:

I. (i) $u+(v+w) = (u+v)+w$, for all $u, v, w \in V$ (Associativity)

(ii) $u+v = v+u$, for all $u, v \in V$ (Commutativity)

(iii) There exists an element $0 \in V$ such that $u+0 = u = 0+u$, for all $u \in V$ (Existence of identity element)

(iv) For each $u \in V$, there exists a unique element $-u \in V$ such that $u+(-u) = 0 = (-u)+u$ (Existence of additive inverse)

II. There is an external composition in V over F called Scalar multiplication.
i.e $\forall \alpha \in F$ and $u \in V \Rightarrow \alpha . u \notin V$
In otherwords V is closed with respect to scalar multiplication.

III. The two compositions, i.e. vector addition and scalar multiplication satisfy the following postulates,

$\forall\ a, b \in F$ and $v, w \in V$

(i) $(a+b)\cdot v = a\cdot v + b\cdot v$

(ii) $a\cdot(b\cdot v) = (ab)\cdot v$

(iii) $a\cdot(v+w) = a\cdot v + a\cdot w$

(iv) $1\cdot(v) = v$.

---

* Vectors in R^n :-

The set of all ordered triples (a, b, c) of real numbers is called Euclidean 3-space and is denoted by R^3 & the set of all n-tuples of real numbers, denoted by R^n, is called Euclidean n-space.

Eg:- The ordered pair $(2, -3)$ belongs to R^2; it is a 2-tuple of dimension two.

The ordered triple $(7, 3, 6)$ belongs to R^3; it is a 3-tuple of dimension three.

Examples or Problems :-

1) Prove that the set of all vectors in a plane over the field of real numbers is a vector space with respect to vector addition and scalar multiplication.

Sol^n:- Let V denotes the set of all coplanar vectors and R be the field of real numbers.

$\therefore$ The elements of V are the ordered pairs (x, y) where $x, y \in R$.

$$V = \{(x, y) : x, y \in R\}$$

I) $(V, +)$ is an abelian group.

i) <u>Associativity</u> :-

We know that for all $u, v, w \in V$

$$(u+v)+w = u+(v+w).$$

ii) <u>Commutativity</u> :-

We know that for all $u, v \in V$

$$u+v = v+u$$

iii) <u>Existence of additive identity</u> :-

For every vector $u \in V$ there exists a zero vector $0 \in V$ such that $u+0 = 0+u = u$.

iv) <u>Existence of additive inverse</u> :-

For every vector $u \in V$ there exists a vector $-u \in V$ such that $u+(-u) = (-u)+u = 0$.

Thus V is an abelian group with respect to vector addition.

II) <u>Scalar multiplication in V.</u>

i) For $u, v \in V$ and $\alpha \in R$ we have

$$\alpha(u+v) = \alpha u + \alpha v$$

ii) For $u \in V$ and $a, b \in R$, we have

$$(a+b)u = au + bu.$$

iii) For $u \in V$ and $a, b \in R$, we have

$$a(bu) = (ab)u$$

iv) For $u \in V$ and $1 \in R$, we have

$$1u = u.$$

Thus set V of coplanar vectors satisfies all the properties of vector addition and scalar multiplication.

$\therefore$ V is a vector space.

Q) Prove that the set C of all complex numbers (i.e the set of all ordered pairs of real numbers) is a vector space over the field R of all real numbers where vector addition is defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \text{ for all } (x_1, x_2), (y_1, y_2) \in C$$

and scalar multiplication is defined by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2), \text{ for all } \alpha \in R.$$

Soln:- We observe that C is closed under vector addition and under scalar multiplication.

I. $(C, +)$ is an abelian group.

(i) Associativity:

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in C$, we have

$$\begin{aligned}(x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] &= (x_1, x_2) + (y_1 + z_1, y_2 + z_2) \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2) \\ &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \\ &= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2)\end{aligned}$$

(ii) Commutativity:

For all $(x_1, x_2), (y_1, y_2) \in C$, we have

$$\begin{aligned}(x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2)\end{aligned}$$

(iii) Existence of identity:

For $(x_1, x_2) \in C$, there exists $(0, 0) \in C$ such that

$$\begin{aligned}(x_1 + x_2) + (0, 0) &= (x_1 + 0, x_2 + 0) \\ &= (x_1, x_2)\end{aligned}$$

$$(0 + 0) + (x_1, x_2) = (0 + x_1, 0 + x_2) = (x_1, x_2)$$

$$\therefore \quad (x_1, x_2) + (0, 0) = (x_1, x_2) = (0, 0) + (x_1, x_2),$$

(iv) Existence of inverse :

For any $(x_1, x_2) \in G$, there exists $(-x_1, -x_2) \in G$ such that
$$(x_1, x_2) + (-x_1, -x_2) = (0, 0) = (-x_1, -x_2) + (x_1, x_2).$$

Thus C is an abelian group with respect to vector addition.

II. Properties of Scalar multiplication in C.

(i)
$$\begin{aligned}\alpha[(x_1, x_2) + (y_1, y_2)] &= \alpha(x_1 + y_1, x_2 + y_2) \\ &= (\alpha(x_1 + y_1), \alpha(x_2 + y_2)) \\ &= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) \\ &= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2) \\ &= \alpha(x_1, x_2) + \alpha(y_1, y_2)\end{aligned}$$

Thus $\alpha[(x_1, x_2) + (y_1, y_2)] = \alpha(x_1, x_2) + \alpha(y_1, y_2)$ for all $(x_1, x_2), (y_1, y_2) \in G, \alpha \in R$.

(ii)
$$\begin{aligned}(a+b)(x_1, x_2) &= ((a+b)x_1, (a+b)x_2) \\ &= (ax_1 + bx_1, ax_2 + bx_2) \\ &= (ax_1, ax_2) + (bx_1, bx_2) \\ &= a(x_1, x_2) + b(x_1, x_2)\end{aligned}$$

Thus $(a+b)(x_1, x_2) = a(x_1, x_2) + b(x_1, x_2)$ for all $(x_1, x_2), (y_1, y_2) \in G$ & $a, b \in R$.

(iii) $a(b(x_1, x_2)) = a(bx_1, bx_2) = (abx_1, abx_2)$

$a(b(x_1, x_2)) = (ab)(x_1, x_2)$

(iv) $1.(x_1, x_2) = (x_1, x_2)$ for all $(x_1, x_2) \in G$

Thus the set G satisfies all the properties of vector space.

Hence G is a vector space over R.

3) Show that the set V of all real valued continuous functions of x defined on interval $[0,1]$ is a vector space over the field R of real numbers with respect to vector addition and scalar multiplication defined by

$$(f_1+f_2)x = f_1(x)+f_2(x), \text{ for all } f_1, f_2 \in V$$

$$(\alpha f_1)x = \alpha f_1(x), \text{ for all } \alpha \in R, f_1 \in V .$$

Sol^n:-

I. $(V,+)$ is an abelian group.

(i) Associativity:

Let $f_1, f_2, f_3 \in V$ be arbitrary.

$$\begin{aligned}[(f_1+f_2)+f_3](x) &= (f_1+f_2)(x)+f_3(x)\\ &= [f_1(x)+f_2(x)]+f_3(x)\\ &= f_1(x)+[f_2(x)+f_3(x)]\\ &= f_1(x)+(f_2+f_3)(x)\\ &= [f_1+(f_2+f_3)](x)\end{aligned}$$

$$(f_1+f_2)+f_3 = f_1+(f_2+f_3).$$

(ii) Commutativity:

Let $f_1, f_2 \in V$ be arbitrary.

$$\begin{aligned}(f_1+f_2)(x) &= f_1(x)+f_2(x)\\ &= f_2(x)+f_1(x)\\ &= (f_2+f_1)(x), \text{ for all } x.\end{aligned}$$

$$f_1+f_2 = f_2+f_1.$$

(iii) Existence of Identity:

Define a function 0 such that $0(x)=0$ (real number), for all $x \in [0,1]$.

Also, 0 is a continuous function and belongs to V

$$(0+f_1)(x) = 0(x)+f_1(x) = 0+f_1(x) = f_1(x)$$

$$(f_1+0)(x) = f_1(x)+0(x) = f_1(x)$$

$$0+f_1 = f_1+0 = f_1$$

Thus, the function 0 defined above is an additive identity element of V.

(iv) <u>Existence of Inverse:</u>

For a function f_1, the function $-f_1$ defined by $(-f_1)(x) = -f_1(x)$, is called additive inverse, as $[f_1 + (-f_1)](x) = f_1(x) + (-f_1)(x)$

$$= f_1(x) - f_1(x) = 0 = 0(x)$$

|||^{ly}, $[-f_1 + f_1](x) = 0(x)$.

Thus the set V is an abelian group under addition.

II. <u>Properties of Scalar Multiplication in V.</u>

(i) For $f_1, f_2 \in V$ and $\alpha \in R$, We have

$$\begin{aligned}[\alpha(f_1 + f_2)]x &= \alpha[(f_1 + f_2)(x)] \\ &= \alpha[f_1(x) + f_2(x)] \\ &= \alpha f_1(x) + \alpha f_2(x) \\ &= (\alpha f_1)x + (\alpha f_2)x \\ &= (\alpha f_1 + \alpha f_2)x\end{aligned}$$

$$\alpha(f_1 + f_2) = \alpha f_1 + \alpha f_2$$

(ii) For $f_1 \in V$ and $a, b \in R$.

$$\begin{aligned}[(a+b)f_1](x) &= (a+b)f_1(x) \\ &= a f_1(x) + b f_1(x) \\ &= (a f_1)x + (b f_1)x \\ &= (a f_1 + b f_1)(x)\end{aligned}$$

$$(a+b)f_1 = a f_1 + b f_1$$

(iii) For $f_1 \in V$ and $a, b \in R$,

$$\begin{aligned}[a(bf_1)]x &= a[(bf_1)x] \\ &= a[b f_1(x)] \\ &= (ab)f_1(x) \\ &= [(ab)f_1]x\end{aligned}$$

$$a(bf_1) = (ab)f_1$$

(iv) For $f_1 \in V$ and $1 \in R$, We have

$$(1.f_1)x = 1.f_1(x) = f_1(x)$$

$$1 f_1 = f_1 .$$

$\therefore$ Thus V satisfies all the properties of a vector space and hence V is a vector space.

---

4) Show that the set of all matrices of the type $m \times n$ where m and n are fixed positive integers is a vector space over R with respect to matrix addition and multiplication of matrix by a scalar (i.e. a real number)

Solⁿ: Let M denotes the set of all matrices of type $m \times n$.

I. $(M, +)$ is an abelian group.

(i) Associativity:

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, $C = [c_{ij}]_{m \times n}$ be three matrices belonging to set M.

$$\begin{aligned}(A+B)+C &= ([a_{ij}] + [b_{ij}]) + [c_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] \\ &= [(a_{ij} + b_{ij}) + (c_{ij})] \\ &= [a_{ij} + (b_{ij} + c_{ij})] \\ &= [a_{ij}] + [b_{ij} + c_{ij}] \\ &= [a_{ij}] + ([b_{ij}] + [c_{ij}]) \\ &= A + (B+C) .\end{aligned}$$

(ii) Commutativity:

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices belonging to set M.

$$A+B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$
$$= [b_{ij} + a_{ij}]$$
$$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$$
$$= B + A.$$

(iii) <u>Existence of Identity</u>:

Let $A = [a_{ij}]_{m\times n} \in M$ be arbitrary.

We know that a zero matrix (null matrix) of type $m\times n$ also belongs to the set M and is denoted by O.

Now, $A + O = [a_{ij}] + [0] = [a_{ij} + 0] = [a_{ij}] = A$

III^{ly}, $O + A = A$.

(iv) <u>Existence of Inverse</u>:

If $A = [a_{ij}]_{m\times n}$ belongs to the set M, then

$-A = [-a_{ij}]_{m\times n}$ also belongs to the set M.

Now, $A + (-A) = [a_{ij}] + [-a_{ij}] = [a_{ij} - a_{ij}]$
$$= [0] = O$$

III^{ly} $(-A) + (A) = O$

Thus the set M is an abelian group under addition.

II. <u>Properties of Scalar multiplication in M.</u>

(i) Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n} \in M$ and $\alpha \in R$, then

$$\alpha(A+B) = \alpha([a_{ij}] + [b_{ij}])$$
$$= \alpha[a_{ij} + b_{ij}]$$
$$= [\alpha(a_{ij} + b_{ij})]$$
$$= [\alpha . a_{ij} + \alpha . b_{ij}]$$ [$\because$ Multiplication is distributive in R]
$$= [\alpha . a_{ij}] + [\alpha . b_{ij}]$$
$$= \alpha[a_{ij}] + \alpha[b_{ij}] = \alpha A + \alpha B.$$

(ii) Let $A = [a_{ij}]_{m\times n} \in M$ and $a, b \in R$, then

$$
\begin{aligned}
(a+b)A &= (a+b)[a_{ij}] \\
&= [(a+b)a_{ij}] \\
&= [a.a_{ij} + b.a_{ij}] \\
&= a[a_{ij}] + b[a_{ij}] \\
&= aA + bA.
\end{aligned}
$$

(iii) Let $A = [a_{ij}]_{m\times n} \in M$ and $a, b \in R$ then

$$
\begin{aligned}
a(bA) &= a(b[a_{ij}]) \\
&= a([ba_{ij}]) \\
&= [(ab)a_{ij}] \\
&= (ab)[a_{ij}] \\
&= (ab)A \\
a(bA) &= (ab)A.
\end{aligned}
$$

(iv) Let $A = [a_{ij}]_{m\times n} \in M$, then

$$1.A = 1.[a_{ij}] = [1.a_{ij}] = [a_{ij}] = A.$$

Thus M satisfies all the properties of vector space and hence M is a vector space over R.

---

5) If V is an abelian group of positive real numbers for multiplication. Define scalar multiplication in V by $a \cdot x = x^a$ where $a \in R$, $x \in V$. Show that V is a vector space over R.

Soln:- Since $V = R^+$ is an abelian group for multiplication.

There are two operations

(i) $\times$, i.e multiplication in R^+

(ii) $\bullet$, i.e Scalar multiplication in R^+

$a \in R, x \in R^+$

$a \cdot x = x^a$

and $x^a \in R^+$ as $x \in R^+$, x^a is always positive.

Eg:- Let $a = -3$

$-3 \cdot x = x^{-3} = \frac{1}{x^3}$ is a positive real number.

Hence x^{-3} is an element ~~of~~ R^+.

Now,

I. $(V, \times)$ is an abelian group.

II. $\forall a \in R, x \in V \Rightarrow a \cdot x = x^a \in V$.

III. (i) $(a+b) \cdot x = x^{a+b} = x^a \times x^b = (a \cdot x) \times (b \cdot x)$

(ii) $a \cdot (x_1 \times x_2) = (x_1 \times x_2)^a$

$= x_1^a \times x_2^a$

$= (a \cdot x_1) \times (a \cdot x_2)$

(iii) $a \cdot (b \cdot x) = (b \cdot x)^a$

$= (x^b)^a = x^{ba}$

$= x^{ab}$ ($\because R$ is commutative for multiplication)

$= ab \cdot (x)$.

(iv) $1 \cdot (x) = x^1 = x$.

Hence $(V, \times, \bullet)$ is a vector space.

Subspaces :-

A non-empty subset W of a vector space $V(F)$ is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplication as for V.

Eg:- Let $W_1 = \{(a,0,0) : a \in R\}$
$W_2 = \{(a,b,0) : a,b \in R\}$.

Here W_1 is a subspace of W_2. Also W_1 and W_2 are subspace of R^3.

* Necessary and Sufficient conditions for a subspace:
(Only statement, no proof).

Theorem 1 : W is a subspace of $V(F)$ iff
i) W is non empty.
ii) W is closed under vector addition.
i.e $\forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.
iii) W is closed under scalar multiplication.
i.e $\forall a \in F$ and $w \in W \Rightarrow a.w \in W$.

Theorem 2 : W is a subspace of $V(F)$ iff
i) W is non empty.
ii) $\forall a,b \in F$ and $v,w \in W \Rightarrow$
$a.v + b.w \in W$.

Problems:-

1) Show that W is a subspace of $V(R)$ where $W = \{f : f(9) = 0\}$

Solⁿ:- Since $0 \in W$ as $0(9) = 0$

So W is non empty set.

Let $f, g \in W$.

i.e $f(9) = 0$ and $g(9) = 0$

then $\forall a, b \in R$

$$(a.f + b.g)9 = a.f(9) + b.g(9) = a.0 + b.0 = 0$$

Hence $a.f + b.g \in W$.

By theorem (2), W is a subspace of $V(R)$.

---

2) Show that W is a subspace of $V(R)$ where $W = \{f : f(2) = f(1)\}$

Solⁿ:- $0 \in W$ since $0(2) = 0 = 0(1)$

hence W is a non empty set.

Let $f, g \in W$ then $f(2) = f(1)$ & $g(2) = g(1)$

then $\forall a, b \in R$.

$$(a.f + b.g)(2) = a.f(2) + b.g(2) = a.f(1) + b.g(1)$$
$$= (a.f + b.g)(1)$$

hence $a.f + b.g \in W$.

So by theorem 2, W is a subspace of $V(R)$.

---

3) Let $V = R^3$ be the Euclidean 3-space. Let $W = \{(x, y, z) : ax + by + cz = 0;\ x, y, z \in R\}$. a, b, c being real numbers. Show that W is a subspace of V.

(OR)

Show that any plane passing through the origin is a subspace of R^3.

Sol^n:- Let $u=(x_1,y_1,z_1)$, $v=(x_2,y_2,z_2)$ be any two elements of W where $x_1,x_2,y_1,y_2,z_1,z_2 \in R$.

Then $ax_1+by_1+cz_1=0$ and $ax_2+bx_2+cx_2=0$.

For $\alpha \in R$, $\alpha u+v = \alpha(x_1,y_1,z_1)+(x_2,y_2,z_2)$

$\alpha u+v = (\alpha x_1, \alpha y_1, \alpha z_1)+(x_2,y_2,z_2)$

$= (\alpha x_1+x_2, \alpha y_1+y_2, \alpha z_1+z_2) \rightarrow (1)$

Where αx_1+x_2, αy_1+y_2, $\alpha z_1+z_2 \in R$.

Now

$a(\alpha x_1+x_2)+b(\alpha y_1+y_2)+c(\alpha z_1+z_2)=$

$\alpha(ax_1+by_1+cz_1)+(ax_2+by_2+cz_2)$

$= \alpha(0)+0=0$. $\rightarrow$ (2).

From (1) and (2), we have

$\alpha u+v \in W$.

Thus, W is a subspace of V.

---

4) Let V be the vector space of all square matrices over R. Determine which of the following are subspaces of V.

(i) $W=\left\{\begin{bmatrix} x & y \\ z & 0 \end{bmatrix} ; x,y,z \in R\right\}$ (ii) $W=\left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x,y \in R\right\}$

(iii) $W=\{A : A \in V \text{ and } A \text{ is singular}\}$ (iv) $W=\{A : A \in V, A^2=A\}$

Solⁿ:- (i) Let $A = \begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$ be any two elements of W.

If $a, b \in R$ then

$$aA + bB = a\begin{bmatrix} x_1 & y_1 \\ z_1 & 0 \end{bmatrix} + b\begin{bmatrix} x_2 & y_2 \\ z_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 & ay_1 \\ az_1 & 0 \end{bmatrix} + \begin{bmatrix} bx_2 & by_2 \\ bz_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ az_1 + bz_2 & 0 \end{bmatrix}$$

which is a matrix of the type $\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$ and

$ax_1 + bx_2,\ ay_1 + by_2,\ az_1 + bz_2 \in R$.

$\therefore\ aA + bB \in W$.

Thus, W is a sub-space of V.

ii) Let $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$ be any two elements of W.

If $a, b \in R$, then

$$aA + bB = a\begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + b\begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & 0 \\ 0 & ax_2 + by_2 \end{bmatrix}$$

which is a matrix of the type $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and

$ax_1 + bx_2,\ ay_1 + by_2 \in R$.

$\therefore\ aA + bB \in W$.

(iii) Here W is the set of singular matrices.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be two square matrices.

Since $|A| = 0$ and $|B| = 0$. therefore $A, B \in W$.

If $a, b \in R$ are non zero, then

$$aA + bB = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Also, $|aA_1 + bA_2| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab \neq 0$, as none of a and b is zero.

$\therefore$ $aA_1 + bA_2 \notin W$.

Thus, W is not a sub-space of V.

(iv) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$

$\therefore$ $A \in W$.

Now $A + A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$$(A+A)^2 = \begin{bmatrix} \cancel{4} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \cancel{A + A}.$$

$\therefore$ $(A + A)$ is not a member of the set W.

$\Rightarrow$ W is not closed under addition.

Hence W is not a sub-space of V.

5) Let V be the vector space of all real valued continuous functions over $\mathbb{R}$. Show that the set W of solutions of differential equations $5\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0$ is a subspace of V.

Solⁿ:- Let $y_1, y_2 \in W$. Then y_1, y_2 are solutions of differential equations.

$$5\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0 \quad \rightarrow (1)$$

i.e $$5\frac{d^2y_1}{dx^2} - 7\frac{d^2y_1}{dx} + 2y_1 = 0 \; ; \; 5\frac{d^2y_2}{dx^2} - 7\frac{d^2y_2}{dx^2} + 2y_2 = 0.$$

Let $a, b \in \mathbb{R}$ be arbitrary.

$$5\frac{d^2}{dx^2}(ay_1 + by_2) - 7\frac{d}{dx}(ay_1 + by_2) + 2(ay_1 + by_2)$$

$$= 5\frac{d^2}{dx^2}(ay_1) - 7\frac{d}{dx}(ay_1) + 2(ay_1) +$$

$$5\frac{d^2}{dx^2}(by_2) - 7\frac{d}{dx}(by_2) + 2(by_2).$$

$$= a\left[5\frac{d^2y_1}{dx^2} - 7\frac{dy_1}{dx} + 2y_1\right] + b\left[5\frac{d^2y_2}{dx^2} - 7\frac{dy_2}{dx} + 2y_2\right]$$

$$= a.0 + b.0 = 0.$$

Thus $ay_1 + by_2$ is a solution of (1) & so $ay_1 + by_2 \in W$

$\therefore$ W is a subspace of V.

Linear Combination :—

Let V be a vector space over a field F and let $v_1, v_2 \dots v_n \in V$.

Any vector of the form $a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n$ in V, where $a_i \in F$ is called a linear combination of $v_1, v_2 \dots v_n$.

Linear Dependence :—

Let V be a vector space over the field F. The vectors $v_1, v_2 \dots v_n$ are said to be linearly dependent over F, if $\exists$ scalars $a_1, a_2 \dots a_n \in F$ not all zero but linear combination is zero.

i.e $a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0$.

but all $a_i \neq 0$, where $i \in N$.

Linear Independence :—

Let V be a vector space over the field F. The vectors $v_1, v_2 \dots v_n$ are said to be linearly independent over F, if $\exists$ scalars $a_1, a_2 \dots a_n \in F$ such that

$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0 \Rightarrow$ all $a_i = 0$ where $i \in N$.

Linear Span :—

Let S be a subset of the vector space V over the field F. The set of all linear combinations of vectors in S is called a linear span of S and is denoted by $L(S)$.

Basis or Base of Vector space V :—

Let V be a vector space over the field F. The set of vectors $\{v_1, v_2 \cdots v_n\}$ is called a basis of V, if

(i) $v_1, v_2 \cdots v_n$ are linearly independent.

(ii) $v_1, v_2 \cdots v_n$ span V. i.e. each vector of V can be uniquely expressed as linear combination of $v_1, v_2 \cdots, v_n$.

Dimension of a vector space V :—

Number of elements in a basis of vector space V is called the dimension of V and is denoted by dim V.

If V contains a basis with n elements then the dim V = n.

Note :-

1) The vector space $\{0\}$ is defined to have dim 0, since empty set ϕ is independent and generates $\{0\}$.

$$\therefore \dim\{0\} = \text{Number of elements in } \phi$$
$$= 0 \;[\text{Since no element is in } \phi]$$

2) When a vector space is not of finite dimension, it is said to be of infinite dimension.

Problems :—

1) Express the vector $v=(1,-2,5)$ as a linear combination of the vectors $v_1=(1,1,1)$, $v_2=(1,2,3)$, $v_3=(2,-1,1)$ in the vector space $R^3(R)$.

Solⁿ :- Let $v = a_1 v_1 + a_2 v_2 + a_3 v_3$; $a_1, a_2, a_3 \in R$

$$(1,-2,5) = a_1(1,1,1) + a_2(1,2,3) + a_3(2,-1,1)$$

$$(1,-2,5) = (a_1+a_2+2a_3,\ a_1+2a_2-a_3,\ a_1+3a_2+a_3)$$

Equating the corresponding elements, we get

$$a_1+a_2+2a_3=1\ ;\ a_1+2a_2-a_3=-2\ ;\ a_1+3a_2+a_3=5$$

Solving above equations, we get

$$a_1=-6,\quad a_2=3,\quad a_3=2$$

Hence $(1,-2,5) = -6(1,1,1)+3(1,2,3)+2(2,-1,1)$.

---

2) Write the vector $v=(1,3,9)$ as a linear combination of the vectors $u_1=(2,1,3)$, $u_2=(1,-1,1)$, $u_3=(3,1,5)$

(Another method of solving)

Solⁿ :- Let $v = xu_1 + yu_2 + zu_3$

$$(1,3,9) = x(2,1,3) + y(1,-1,1) + z(3,1,5)$$

$$\Rightarrow \quad 2x+y+3z=1\ ;\quad x-y+z=3\ ;\quad 3x+y+5z=9$$

Consider $AX=B$.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

Augmented matrix,

$$[A:B] = \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 1 & -1 & 1 & : & 3 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 2 & 1 & 3 & : & 1 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 4 & 2 & : & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 1 & 1 & : & 5 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & +5 \\ 0 & 3 & 1 & : & -5 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & 5 \\ 0 & 0 & -2 & : & -20 \end{bmatrix}$$

$\therefore$ Rank $[A]$ = Rank $[A:B]$ = 3 = Number of unknowns

$\therefore$ The system of linear equations is consistent and Possesses a unique solⁿ.

$x + y + z = 3$; $y + z = 5$; $-2z = -20$

$\therefore$ $z = 10$, $y = -5$, $x = -12$

$\therefore\ (1,3,9) = -12(2,1,3) - 5(1,-1,1) + 10(3,1,5)$.

3) Write the vector $v = (4,2,1)$ as a linear combination of the vectors $u_1 = (1,-3,1)$, $u_2 = (0,1,2)$, $u_3 = (5,1,37)$.

Sol^n:- Let $v = xu_1 + yu_2 + zu_3$

$$(4,2,1) = x(1,-3,1) + y(0,1,2) + z(5,1,37)$$

$$\Rightarrow \begin{aligned} x + 0y + 5z &= 4 \\ -3x + y + z &= 2 \\ x + 2y + 37z &= 1 \end{aligned}$$

Let $AX = B$. where $A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 1 \\ 1 & 2 & 37 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Consider, $[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ -3 & 1 & 1 & 2 \\ 1 & 2 & 37 & 1 \end{array}\right]$

$R_2 \to R_2 + 3R_1$
$R_3 \to R_3 - R_1$

$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 2 & 32 & -3 \end{array}\right]$

$R_3 \to R_3 - 2R_2$

$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 0 & 0 & -31 \end{array}\right]$

Rank $[A] = 2 \neq$ Rank $[A:B] = 3$.

Since Rank $[A] \neq$ Rank $[A:B]$. Hence the system of linear equations is inconsistent.

i.e sol^n does not exist.

$\therefore$ v cannot be expressed as linear combination of the vectors u_1, u_2, u_3.

4) For what value of k (if any) the vector $v=(1,-2,k)$ can be expressed as a linear combination of vectors $v_1=(3,0,-2)$ and $v_2=(2,-1,-5)$ in $R^3(R)$.

Sol^n:- Since vector $v=(1,-2,k)$ is a linear combination of $v_1=(3,0,-2)$ and $v_2=(2,-1,-5)$; therefore there exist scalars a and b such that

$$v = av_1 + bv_2$$
$$(1,-2,k) = a(3,0,-2) + b(2,-1,-5)$$
$$(1,-2,k) = (3a+2b\ ,\ -b\ ,\ -2a-5b)$$

Equating the corresponding elements,

$3a+2b=1$, $-b=-2$, $-2a-5b=k$

$b=2$

$3a+4=1$ $\qquad -2(-1)-5(2)=k$

$a=-1$ $\qquad 2-10=k$

$$\boxed{\therefore\ k=-8.}$$

---

5) Find a condition on a, b, c so that $w=(a,b,c)$ is a linear combination of $u=(1,-3,2)$ and $v=(2,-1,1)$ in R^3 so that $w \in \text{Span}(u,v)$.

Sol^n:- Let $w = xu + yv$

$$(a,b,c) = x(1,-3,2) + y(2,-1,1)$$
$$x+2y=a\ ;\ -3x-y=b\ ;\ 2x+y=c$$

Let $AX=B$.

Consider $[A:B] = \begin{bmatrix} 1 & 2 & : & a \\ -3 & -1 & : & b \\ 2 & 1 & : & c \end{bmatrix}$

$R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - 2R_1$

$$[A:B] = \left[\begin{array}{cc:c} 1 & 2 & a \\ 0 & 5 & b+3a \\ 0 & -3 & c-2a \end{array}\right]$$

$$R_2 \to \frac{R_2}{5}$$

$$[A:B] = \left[\begin{array}{cc:c} 1 & 2 & a \\ 0 & 1 & \frac{b+3a}{5} \\ 0 & -3 & c-2a \end{array}\right]$$

$$R_3 \to R_3 + 3R_2$$

$$[A:B] = \left[\begin{array}{cc:c} 1 & 2 & a \\ 0 & 1 & \frac{b+3a}{5} \\ 0 & 0 & \frac{5c-a+3b}{5} \end{array}\right]$$

Rank $[A] = 2$

The system of linear equations will be consistent if rank $[A:B] = 2$

So, $\frac{5c-a+3b}{5} = 0 \Rightarrow a-3b-5c=0$.

i.e. w is linear combination of u and v iff $a-3b-5c=0$.

---

6) Express the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of

$B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Soln :- Let $A = a_1 B + a_2 C + a_3 D$; $a_1, a_2, a_3 \in R$. $\to (1)$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a_1+a_2+a_3 & a_1+a_2-a_3 \\ -a_2 & -a_1 \end{bmatrix}$$

Equating the corresponding elements,

$$a_1+a_2+a_3=3 \; ; \; a_1+a_2-a_3=-1 \; ; \; -a_2=1 \; ; \; -a_1=-2$$

Solving the above equations,

$$a_1=2, \; a_2=-1, \; a_3=2 .$$

$\therefore$ (1) $\Rightarrow$

$$A = 2B - C - 2D .$$

(or)

$$\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is the required linear combination of A.

---

7) Express the matrix $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ as a linear combination of the matrices $A=\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $C=\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

Soln:- Let $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = a_1\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + a_3\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$;

$a_1, a_2, a_3 \in R$.

$$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 0+0+2a_3 & -3a_1+3a_3 \\ 2a_1+2a_2 & a_2+5a_3 \end{bmatrix}$$

Equating the corresponding elements

$2 = 2a_3 \rightarrow$ (1)

$0 = -3a_1+3a_3 \rightarrow$ (2)

$4 = 2a_1+2a_2 \rightarrow$ (3)

$-5 = a_2+5a_3 . \rightarrow$ (4)

$2 = 2a_3 \Rightarrow a_3 = 1$

$0 = -3a_1 + 3a_3 \Rightarrow a_1 = 1$

$4 = 2a_1 + 2a_2 \Rightarrow a_2 = 1$

But $a_1=1, a_2=1, a_3=1$ do not satisfy eq^n (4)

Thus equations (1), (2) & (3) have no solution.

Hence, the given matrix cannot be expressed as a linear combination of A, B, C

Note :-

* Two vectors v_1 and v_2 are linearly dependent iff one of them is a multiple of the other.

Eg:- Determine whether or not the vectors v_1, v_2 are linearly dependent.

i) $v_1 = (1, 3, 9)$ $v_2 = (2, 4, 1)$.

→ No vector is the multiple of the other hence v_1 & v_2 are not linearly dependent.

ii) $v_1 = (3, 4)$ $v_2 = (6, 8)$.

→ $v_2 = 2v_1$ hence v_1 and v_2 are linearly dependent vectors.

---

8) Determine whether the vectors $v_1 = (1, 2, 3)$ $v_2 = (3, 1, 7)$ and $v_3 = (2, 5, 8)$ are linearly dependent or linearly independent.

Soln:- $x v_1 + y v_2 + z v_3 = 0$

$x(1, 2, 3) + y(3, 1, 7) + z(2, 5, 8) = (0, 0, 0)$

$x + 3y + 2z = 0$; $2x + y + 5z = 0$; $3x + 7y + 8z = 0$.

Let $AX = 0$ where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$R_2 \to -R_2$
$R_3 \to -\frac{R_3}{2}$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Rank $[A] = 3 =$ Number of unknowns.

$\Rightarrow$ Homogeneous system of linear equations possesses unique solution.

i.e $x=0,\ y=0,\ z=0$.

So vectors v_1, v_2, v_3 are linearly independent.

9) Determine whether the vectors $v_1 = (1,9,3)$, $v_2 = (2,5,4)$, $v_3 = (0,0,0)$ are linearly dependent or linearly independent.

Solⁿ:- vector v_1, v_2, v_3 are linearly dependent as linearly independent set of vectors can not contain zero vector.

---

10) Determine whether the vectors $v_1 = (1,4,9)$, $v_2 = (3,1,4)$, $v_3 = (9,3,12)$ are linearly dependent or linearly independent.

Solⁿ:- $xv_1 + yv_2 + zv_3 = 0$.

$x(1,4,9) + y(3,1,4) + z(9,3,12) = (0,0,0)$

$x+3y+9z=0$; $4x+y+3z=0$; $9x+4y+12z=0$.

Consider $AX=0$. where

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 9R_1$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & -23 & -69 \end{bmatrix}$$

$R_2 \to -\frac{R_2}{11}$

$R_3 \to -\frac{R_3}{29}$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $[A] = 2 < 3$ (Number of unknowns)

So one unknown is constant and hence not always zero.

So the vectors are linearly dependent.

[OR]

$V_3 = 3V_2$

Vectors v_1, v_2, V_3 are linearly dependent.

---

11) If u, v, w are linearly independent vectors in $V(F)$, where F is the field of complex numbers, then $\{u+v, v+w, w+u\}$ is a linearly independent set of vectors.

Soln:- Let $a(u+v) + b(v+w) + c(w+u) = 0$, where $a, b, c \in F$

$$(a+c)u + (a+b)v + (b+c)w = 0 \to (1)$$

Since u, v, w are linearly independent.

$\therefore$ (1) $\Rightarrow$ $a+c=0$

$a+b=0$

$b+c=0$.

$\therefore$ $a=0, b=0, c=0$.

12) Let V be the vector space of all 2×3 matrices over R. S.T the matrices $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ form a linearly independent set.

Soln:- Let $aA + bB + cC = 0$ where $a, b, c \in R$.

$$\Rightarrow a\begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + b\begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + c\begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+b+4c & a+b-c & -a-3b+2c \\ 3a-2b+c & -2a+0b-2c & 4a+5b+3c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By equating the corresponding elements

$$\left.\begin{aligned} 2a+b+4c &= 0 \\ a+b-c &= 0 \\ -a-3b+2c &= 0 \end{aligned}\right\} \rightarrow (1) \qquad \left.\begin{aligned} 3a-2b+c &= 0 \\ -2a-2c &= 0 \\ 4a+5b+3c &= 0. \end{aligned}\right\} \rightarrow (2)$$

On solving the system (1) of equations, we observe that the only solution is $a=0, b=0, c=0$. Clearly this solution also satisfies the system (2) of equations.

Hence the given set of matrices is linearly independent.

---

13) Show that the vectors $(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0)$ and $(2,1,1,6)$ are linearly dependent.

14) Determine whether or not each of the following forms a basis, $x_1 = (1, 2, 3)$, $x_2 = (3, -5, 6)$ in R^3.

Soln:- Basis in R^3 must contain exactly 3 elements therefore x_1, x_2 does not form a basis in R^3.

15) Determine whether or not each of the following form a basis $x_1 = (1,2,9)$, $x_2 = (2,-3,4)$, $x_3 = (1,3,7)$, $x_4 = (2,4,8)$ in R^3.

Soln:- Basis in R^3 must contain exactly 3 elements.

$\therefore$ x_1, x_2, x_3, x_4 does not form a basis in R^3.

---

16) Determine whether or not each of the following forms a basis, $x_1 = (2,2,1)$, $x_2 = (1,3,7)$, $x_3 = (1,2,2)$ in R^3.

Soln:- Three vectors in R^3 form a basis iff they are linearly independent.

$$x(2,2,1) + y(1,3,7) + z(1,2,2) = (0,0,0)$$

$$2x + y + z = 0; \quad 2x + 3y + 2z = 0; \quad x + 7y + 2z = 0.$$

Let $AX = 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 7 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & -13 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to -R_2 \\ R_3 \to -R_3 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 13 & 3 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{2}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 1 & 1/2 \end{bmatrix}_{R_3 \leftrightarrow R_2} = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1/2 \\ 0 & 11 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - 11R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -7/2 \end{bmatrix}$$

Rank [A] = 3 = Number of unknowns. So unique solution and solution is

$x = 0, \ y = 0, \ z = 0$.

Hence vectors x_1, x_2, x_3 are linearly independent and hence form a basis.

17) Let W be the subspace of R^5, spanned by $x_1 = (1, 2, -1, 3, 4)$, $x_2 = (2, 4, -2, 6, 8)$, $x_3 = (1, 3, 2, 2, 6)$ $x_4 = (1, 4, 5, 1, 8)$, $x_5 = (2, 7, 3, 3, 9)$.

Find a subset of vectors which forms a basis of W.

Soln:- $$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$R_5 \to R_5 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3,\ R_5 \to R_5 - 3R_3$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

Number of non zero rows = 3

$\therefore$ dim $W = 3$ and $\{x_1, x_3, x_5\}$ forms a basis in W.

---

18) V is a vector space of polynomials over R. Find a basis and dimension of the subspace W of V, spanned by the polynomials.

$x_1 = t^3 - 2t^2 + 4t + 1$, $x_2 = 2t^3 - 3t^2 + 9t - 1$

$x_3 = t^3 + 6t - 5$, $x_4 = 2t^3 - 5t^2 + 7t + 5$.

Soln:— $A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 2R_1$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 + R_2$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non Zero rows of Echelon matrix forms a basis.

Number of non Zero rows = 2

$\therefore$ dim $W = 2$ and $\{x_1, x_2\}$ forms a basis in W.

Linear Transformations :–

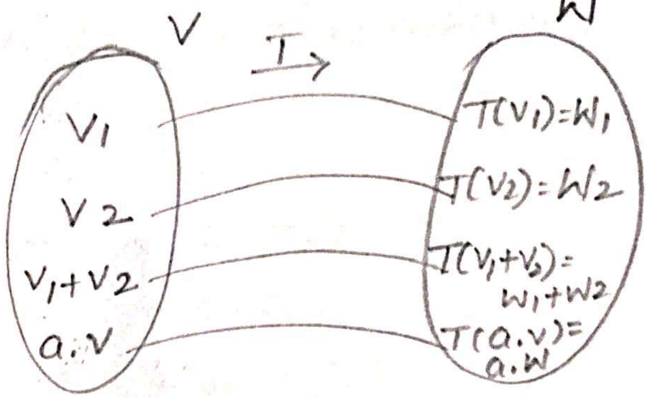
Let V and W be any two subspaces over the field F. A mapping T from V to W is called a linear transformation if,

i) $v_1, v_2 \in V$

$T(v_1+v_2) = T(v_1) + T(v_2)$

ii) $\forall a \in F$ and $\forall v \in V$

$T(a.v) = a.T(v)$



Problems :–

1) Show that the function $T: R^2 \to R^3$ given by $T(x,y) = (x+y, x-y, y)$ is a linear transformation.

Solⁿ:– Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ be two vectors belonging to R^2.

$\therefore T(u+v) = T(x_1+x_2, y_1+y_2)$

$= (x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2, y_1+y_2)$

$= \{(x_1+y_1)+(x_2+y_2), (x_1-y_1)+(x_2-y_2), (y_1+y_2)\}$

$= (x_1+y_1, x_1-y_1, y_1) + (x_2+y_2, x_2-y_2, y_2)$

$= T(u) + T(v)$

Also for $a \in R$ and $u \in R^2$, we have

$T(au) = T(ax_1, ay_1)$

$= (ax_1+ay_1, ax_1-ay_1, ay_1)$

$= a(x_1+y_1 \quad x_1-y_1 \quad y_1)$

$= a\,T(u)$

$\therefore$ T is a linear transformation.

2) Which of the following functions are linear transformations?

i) $T: R^3 \to R^3$ defined by $T(x,y,z) = (y,-x,-z)$

ii) $T: R^3 \to R^2$ defined by $T(x,y,z) = (2x-3y, 7y+2z)$.

Soln:- i) Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in R^3$ be arbitrary.

$T(u) = T(x_1, y_1, z_1) = (y_1, -x_1, -z_1)$

$T(v) = T(x_2, y_2, z_2) = (y_2, -x_2, -z_2)$

Now, $T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (y_1+y_2, -x_1-x_2, -z_1-z_2)$

$= (y_1, -x_1, -z_1) + (y_2, -x_2, -z_2)$

$= T(u) + T(v)$

Also, for any scalar $a \in R$,

$T(au) = T(ax_1, ay_1, az_1)$

$= (ay_1, -ax_1, -az_1)$

$= a(y_1, -x_1, -z_1)$

$= aT(u)$

Hence, T is a linear transformation.

ii) Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2) \in R^3$ be arbitrary.

$T(u) = T(x_1, y_1, z_1) = (2x_1 - 3y_1, 7y_1 + 2z_1)$

$T(v) = T(x_2, y_2, z_2) = (2x_2 - 3y_2, 7y_2 + 2z_2)$

$\therefore T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= (2x_1 + 2x_2 - 3y_1 - 3y_2, 7y_1 + 7y_2 + 2z_1 + 2z_2)$

$= \{(2x_1 - 3y_1) + (2x_2 - 3y_2), (7y_1 + 2z_1) + (7y_2 + 2z_2)\}$

$= (2x_1 - 3y_1, 7y_1 + 2z_1) + (2x_2 - 3y_2, 7y_2 + 2z_2)$

$= T(u) + T(v)$

Also, for any scalar $a \in R$

$T(au) = T(ax_1, ay_1, az_1)$

$= (2ax_1 - 3ay_1, 7ay_1 + 2az_1)$

$= a(2x_1 - 3y_1, 7y_1 + 2x_1)$

$= aT(u)$.

Hence, T is a linear transformation.

3) Let $I: V(R) \to V_2(R)$ be a mapping $f(x) = (3x, 5x)$. Show that f is a linear transformation.

Soln:- i) $f(x_1+x_2) = (3(x_1+x_2), 5(x_1+x_2))$

$= (3x_1 + 3x_2, 5x_1 + 5x_2)$

$= (3x_1 \cancel{+} 5x_1) + (3x_2 \cancel{+} 5x_2)$

$= f(x_1) + f(x_2)$

ii) $f(a \cdot x) = (3(a \cdot x), 5(a \cdot x))$

$= (3ax, 5ax)$

$= a(3x, 5x)$

$= a \cdot f(x)$.

Hence f is a linear transformation.

4) Show that the translation mapping $f: V_2(R) \to V_2(R)$ defined by $f(x, y) = (x+6, y+2)$ is not linear.

Soln:- $f(x_1+x_2, y_1+y_2) = (x_1+x_2+6, y_1+y_2+2) \to (1)$

$f(x_1, y_1) + f(x_2, y_2) = (x_1+6, y_1+2) + (x_2+6, y_2+2)$

$= (x_1+x_2+12, y_1+y_2+4) \to (2)$

Equation (1) $\neq$ Equation (2).

Hence f is not a linear transformation.

* Kernel of T and Image of T:

Let T be a linear transformation from V to W

Kernel of T or Ker $(T) = \{v \in V : T(v) = 0, 0 \in W\}$

Remark :- Kernel of T is also known as null space of T.

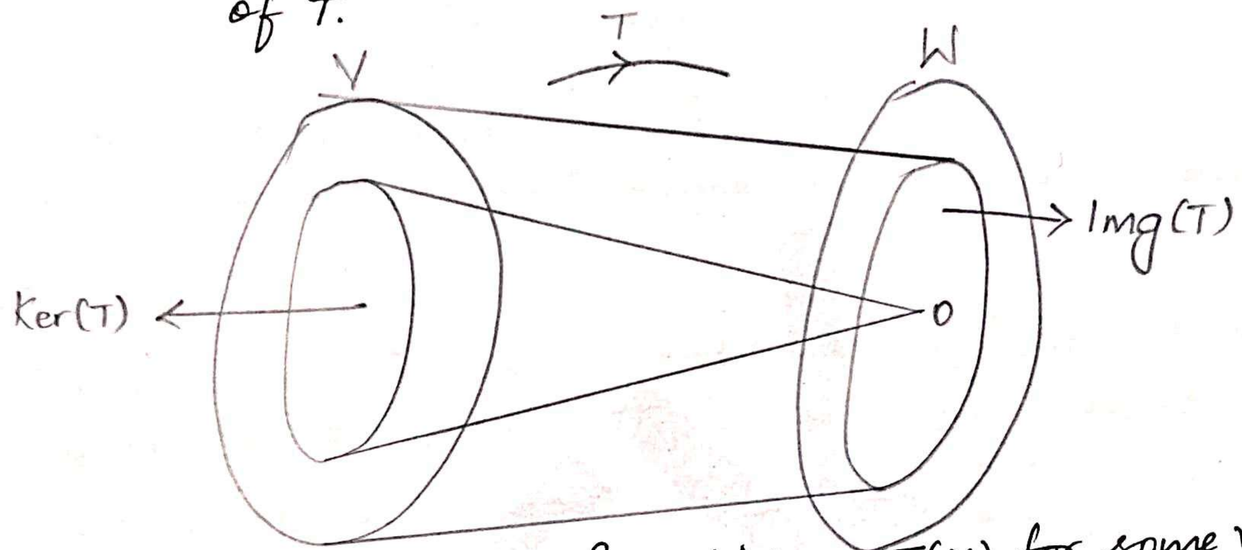


Image of T or $Img(T) = \{w \in W ; w = T(v) \text{ for some } v \in V\}$

Algebra of Linear Transformations :-

* Sum of Linear transformations:-

Let $T_1 : U \to V$ and $T_2 : U \to V$ be two transformations where U and V are two vectors spaces over the field F. We define the sum of T_1 and T_2 by $T_1 + T_2 : U \to V$, such that $(T_1 + T_2)u = T_1(u) + T_2(u)$, $u \in U$.

* Scalar multiplication of Linear transformations:-

If $a \in F$, the function aT, i.e product of a linear transformation $T : U \to V$ with a, and defined by $(aT) : U \to V$, such that $(aT)u = a(T(u))$ for all $u \in U$ is a linear transformation from U into V.

* Composition (Product) of two Linear transformations:

Let U, V, W be three vector spaces over a field F. We define $T_2 T_1$, called the composition or product of T_2 and T_1, by $(T_2 T_1)(u) = T_2(T_1(u))$.

$T_2 T_1$ is also denoted by $(T_2 \circ T_1)$.

Problems:—

1) Let the linear transformations $T_1 : R^3 \to R^2$ such that $T_1(x, y, z) = (4x, 3y - 2z)$ and $T_2 : R^2 \to R^2$ such that $T_2(x, y) = (-2x, y)$. Compute $T_1 T_2$ and $T_2 T_1$.

Soln:— As the range of T_2 is not contained in the domain of T_1

$\therefore$ $T_1 T_2$ is not defined.

Now, $T_2 T_1$ is defined as the range of T_1 is contained in the domain of T_2.

$$
\begin{aligned}
T_2 T_1 (x, y, z) &= T_2 (T_1(x, y, z)) \\
&= T_2 (4x, 3y - 2z) \\
&= (-8x, 3y - 2z).
\end{aligned}
$$

---

2) Illustrate with the help of an example that there exist linear transformations $T_1 : R^2 \to R^2$ and $T_2 : R^2 \to R^2$ such that $T_2 T_1 = 0$ but $T_1 T_2 \neq 0$.

Soln:- Let us define $T_1 : R^2 \to R^2$ by

$$T_1(x, y) = (0, 4x)$$

& $T_2 : R^2 \to R^2$ by $T_2(x, y) = (x, 0)$

$T_1 T_2$ and $T_2 T_1$ both are defined.

$$(T_2T_1)(x,y) = T_2(T_1(x,y)) = T_2(0,4x)$$
$$= (0,0) = 0(x,y)$$
$$T_2T_1 = 0.$$

Also $(T_1T_2)(x,y) = T_1(T_2(x,y)) = T_1(x,0)$
$$= (0,4x) \neq 0(x,y)$$
$$T_1T_2 \neq 0.$$

Matrix of a Linear transformation:—

Let $T: U \to V$ be the linear transformation, where U and V are vector spaces over field F.

Let $B = \{u_1, u_2 ---- u_n\}$ and $B' = \{v_1, v_2 --- v_m\}$ be ordered basis for the finite dimensional vector spaces U and V respectively.

Since $T(u_1), T(u_2) ---- T(u_n) \in V$ and $\{v_1, v_2 --- v_m\}$ spans V, each $T(u_i)$ can be expressed as a linear combination of the vectors $v_1, v_2 --- v_m$.

Let $T(u_1) = a_{11}v_1 + a_{21}v_2 + ---- + a_{m1}v_m$
$T(u_2) = a_{12}v_1 + a_{22}v_2 + ---- + a_{m2}v_m$

$- \; - \; - \; - \; - \; - \; - \; - \; -$

$T(u_n) = a_{1n}v_1 + a_{2n}v_2 + ---- + a_{mn}v_m$ where $a_{ij} \in F$.

The coefficient matrix of this system of equations is

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} & ---- & a_{m1} \\ a_{12} & a_{22} & a_{32} & ---- & a_{m2} \\ - & - & - & ---- & - \\ a_{1n} & a_{2n} & a_{3n} & --- & a_{mn} \end{bmatrix}$$

The transpose of this matrix is a matrix representation of T, called matrix of T with repect to ordered basis B and B' (or matrix associated with T w.r.to B and B']

It is denoted by [T:B, B'] and is given by

$$[T:B,B'] = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ \vdots & \vdots & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix}$$

Problems:-

1) Find the matrix representing the transformation $T: R^2 \to R^3$ given by $T(x_1, x_2) = (3x_1 - x_2,\ 2x_1 + 4x_2,\ 5x_1 - 6x_2)$ relative to the standard basis of R^2 and R^3.

Soln:- The ordered standard basis of R^2 is $B = \{(1,0),(0,1)\}$ and that of R^3 is

$B' = \{(1,0,0),(0,1,0),(0,0,1)\}$.

$\therefore T(x_1, x_2) = (3x_1 - x_2,\ 2x_1 + 4x_2,\ 5x_1 - 6x_2)$

$T(1,0) = (3,2,5)$ & $T(0,1) = (-1,4,-6)$

$T(1,0) = (3,2,5) = 3(100) + 2(010) + 5(001)$

$T(0,1) = (-1,4,-6) = -1(100) + 4(010) + (-6)(001)$

Hence, matrix of the transformation is $\begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 5 & -6 \end{bmatrix}$

---

2) Find the matrix representing the transformation $T: R^3 \to R^4$ defined by $T(x,y,z) = (x+y+z,\ 2x+z,\ 2y-z,\ 6y)$ relative to the standard basis of R^3 and R^4.

Sol^n:- We know that ordered standard basis of R^3 is
$B = \{(1,0,0), (0,1,0), (0,0,1)\}$ and
R^4 is $B' = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

Since $T(x,y,z) = (x+y+z,\ 2x+z,\ 2y-z,\ 6y)$

$T(1,0,0) = (1,2,0,0)$

$T(0,1,0) = (1,0,2,6)$

$T(0,0,1) = (1,1,-1,0)$

Let $e_1 = (1,0,0,0)$ $e_2 = (0,1,0,0)$ $e_3 = (0,0,1,0)$ $e_4 = (0,0,0,1)$

$$\therefore \left.\begin{aligned} T(1,0,0) &= (1,2,0,0) = 1e_1 + 2e_2 + 0e_3 + 0e_4 \\ T(0,1,0) &= (1,0,2,6) = 1e_1 + 0e_2 + 2e_3 + 6e_4 \\ T(0,0,1) &= (1,1,-1,0) = 1e_1 + 1e_2 - 1e_3 + 0e_4 \end{aligned}\right\} \rightarrow (1)$$

Matrix of T relative to B and B' is transpose of matrix of coefficients in above system of eq^n (1)

i.e, $$[T:B,B'] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}.$$

---

3) Find the linear map $T: R^2 \rightarrow R^3$ whose matrix is
$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$, relative to the ordered basis $B: \{(1,1)\ (0,2)\}$ and
$B' = \{(0,1,1), (1,0,1), (1,1,0)\}$ for R^3.

Sol^n:- Here $[T:B,B'] = \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$

$T(1,1) =$ Linear Combination of vectors of B' using scalars of first column of $[T:B,B']$

$= 1(0,1,1) - 2(1,0,1) + 0(1,1,0) = (-2,1,-1)$

$|||^{ly}$ $T(0,2) = -1(0,1,1) + 3(1,0,1) + 1(1,1,0) = (4,0,2)$

Let $(x,y) \in R^2$ be arbitrary.

$$(x,y) = a(1,1) + b(0,2)$$

To find a and b:—

$$(x,y) = (a, a+2b) \Rightarrow a = x$$

and $a+2b = y \Rightarrow b = \frac{y-x}{2}$

Using these values,

$$(x,y) = x(1,1) + \frac{y-x}{2}(0,2)$$

Applying T on both sides,

$$T(x,y) = x\,T(1,1) + \left(\frac{y-x}{2}\right)T(0,2)$$

$$= x(-2,1,-1) + \left(\frac{y-x}{2}\right)(4,0,2)$$

$$= \left[-2x + \left(\frac{y-x}{2}\right)4,\ x+0,\ -x + \left(\frac{y-x}{2}\right)2\right]$$

$= (2y-4x,\ x,\ y-2x)$ which is the required transformation.

Change of coordinates or change of Basis Matrix or Transition Matrix :—

Let $B_1 = \{u_1, u_2 \ldots u_n\}$ be a basis of n-dimensional vector space V and $B_2 = \{v_1, v_2 \ldots v_n\}$ be another basis of V. Then, each element in B_2 can be expressed as a linear combination of the vectors in basis B_1.

Let $v_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$
$v_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$
$\vdots$
$v_n = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Then the transpose of the matrix of coefficients in above system of equations is said to be the transition matrix or change of basis matrix from the old basis B_1 to the new basis B_2. It is denoted by P and written as

$$P = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ - & - & - & - \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

Since the vectors in B_2 are linearly independent, so the matrix P is invertible. Its inverse P^{-1} is the transition matrix from the basis B_2 to the basis B_1.

Problems:-

1) Consider the following basis of R^2:
$E = \{e_1, e_2\} = \{(1\,0), (0\,1)\}$ and $S = \{u_1, u_2\} = \{(1,3), (1,4)\}$

(a) Find the change-of-basis matrix P from the usual basis E to S.

(b) Find the change-of-basis matrix Q from S back to E.

(c) Find the coordinate vector $[v]$ of $v = (5, -3)$ relative

(a) Since E is the usual basis of R^2.

$$\therefore \quad P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

(b) Find P^{-1}, $P^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

(c) Let $[v]_S = P^{-1}[v]_E$.

We have $[v]_E = [5, -3]^T$

$$[v]_S = P^{-1}[v]_E = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ -18 \end{bmatrix}.$$

Q2) The vectors $u_1 = (1, 2, 0)$, $u_2 = (1, 3, 2)$, $u_3 = (0, 1, 3)$ form a basis S of R^3. Find:

a) The change of basis matrix P from the usual basis $E = \{e_1, e_2, e_3\}$ of R^3 to the basis S.

b) The change of basis matrix Q from the above basis S back to the usual basis E of R^3.

Soln :-

(a) Since E is a usual basis of R^3.

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

(b) Find P^{-1}, $P^{-1} = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$

3) Find the Co-ordinates of vector $(1,1,1)$ relative to basis $(1,1,2), (2,2,1), (1,2,2)$.

Soln:- Let $(1,1,1) = a(1,1,2) + b(2,2,1) + c(1,2,2)$

$$a+2b+c=1 \rightarrow (1)$$
$$a+2b+2c=1 \rightarrow (2)$$
$$2a+b+2c=1 \rightarrow (3)$$

Subtracting (1) from (2), we have $c=0$

Putting $c=0$ in (2) and (3), we get

$$a+2b=1 \rightarrow (4)$$
$$2a+b=1 \rightarrow (5)$$

Solving (4) and (5), we get $a=b=\frac{1}{3}$.

$\therefore$ Coordinates of $(1,1,1)$ relative to given basis are $\left(\frac{1}{3}, \frac{1}{3}, 0\right)$.

---

4) Find the co-ordinates of the following vectors relative to the basis $v_1 = (1,1,2)$, $v_2 = (2,2,1)$, $v_3 = (1,2,2)$.

(i) $(1,0,1)$ soln:- $\left(\frac{4}{3}, \frac{1}{3}, -1\right)$

(ii) $(1,1,0)$ soln:- $\left(-\frac{1}{3}, \frac{2}{3}, 0\right)$

Rank and nullity of a linear Operator:—

Rank of T:—

Dimension of the $Img(T)$ is known as rank of T.

Nullity of T:—

Dimension of the $Ker(T)$ is known as nullity of T.

Rank-nullity Theorem:—

Let V and W be vector spaces over the field F and let T be a linear transformation from V to W. If V is a finite dimensional then

$$\text{rank}(T) + \text{nullity of }(T) = \dim(V).$$

Problems:—

1) For the linear transformation $T: R^2 \to R^3$ such that $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, -x_1)$, find a basis and dimension of its range space and its null space. Also verify that $\text{rank}(T) + \text{nullity}(T) = \dim R^2$.

Solⁿ:- i) To find ~~all~~ null space of T and its dimension:

By defⁿ, null space $= N(T) = \{u \in R^2 : T(u) = 0 \in R^3\}$

Let $u = (x_1, x_2) \in R^2$ be an arbitrary element of null space.

$$T(u) = 0$$

$$T(x_1, x_2) = 0$$

$$(x_1 - x_2, x_2 - x_1, -x_1) = 0$$

$\Rightarrow \quad x_1 - x_2 = 0$

$x_2 - x_1 = 0$

$-x_1 = 0$

On solving the above equations, We get

$x_1 = 0, \; x_2 = 0$.

Thus, the only member of null space is a zero vector $\in R^2$

i.e $N(T) = \{0\}$.

$\therefore$ Nullity of $T = \dim(N(T)) = 0$.

ii) To find range space of T and its dimension:

The range space consists of ordered triples $(x_1 - x_2, \; x_2 - x_1, \; -x_1)$ for $(x_1, x_2) \in R^2$ such that

$$T(x_1, x_2) = (x_1 - x_2, \; x_2 - x_1, \; -x_1) \rightarrow (1)$$

Let v be any element of $R(T)$.

Thus there exist $(x_1, x_2) \in R^2$ such that $v = T(x_1, x_2)$

$\Rightarrow \; (x_1, x_2) = x_1(1,0) + x_2(0,1)$

$= x_1 e_1 + x_2 e_2$ where $e_1 = (1,0), \; e_2 = (0,1)$

$T(x_1, x_2) = T(x_1 e_1 + x_2 e_2)$

$= x_1 T(e_1) + x_2 T(e_2) \longrightarrow (2)$

$T(e_1) = T(1,0) = (1-0, 0-1, -1) = (1,-1,-1)$

$T(e_2) = T(0,1) = (0-1, 1-0, 0) = (-1,1,0)$

Putting the values of $T(e_1), T(e_2)$ in (2)

$T(x_1, x_2) = x_1(1,-1,-1) + x_2(-1,1,0)$

$$v = x_1(1,-1,-1) + x_2(-1,1,0) \rightarrow (3)$$

Since $v \in R(T)$ is arbitrary,

(iii) Let $a(1,-1,-1) + b(-1,1,0) = 0$ for $a, b \in F$

$\Rightarrow (a-b, -a+b, -a) = (0,0,0)$

$\Rightarrow a-b=0, \; b-a=0, \; -a=0$

$\Rightarrow a=b=0.$

Thus $\{(1,-1,-1),(-1,1,0)\}$ is linearly independent, a

$\therefore$ It is a basis set of $R(T)$.

$\therefore$ Dimension of $R(T) = 2$

Also dimension of $R^2 = 2$

$\therefore$ Rank (T) + nullity $(T) = 2+0 = 2 = \dim R^2$.

---

2) Find a linear transformation $T: R^3 \to R^3$ whose rangespace is spanned by the vectors $(1,2,3),(4,5,6)$

Sol^n:- We know that $\{e_1, e_2, e_3\}$ is a standard basis of R^3, where $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$ and Since R^3 is a three dimensional vector space and $\{e_1, e_2, e_3\}$ is a basis of R^3, there exist a unique linear transformation T such that

$T(e_1) = (1,2,3)$

$T(e_2) = (4,5,6)$

$T(e_3) = (0,0,0)$ (Assuming $T(0,0,1) = (0,0,0)$).

Also $R(T)$ is spanned by $\{(1,2,3),(4,5,6)\}$

i.e by $\{(1,2,3),(4,5,6),(0,0,0)\}$ or by $\{T(e_1), T(e_2), T(e_3)\}$

Now, for each $(x_1, x_2, x_3) \in R^3$,

$$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$
$$= x_1 e_1 + x_2 e_2 + x_3 e_3$$

$$T(x_1,x_2,x_3) = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$$
$$= x_1(1,2,3) + x_2(4,5,6) + x_3(0,0,0)$$
$$= (x_1+4x_2,\ 2x_1+5x_2,\ 3x_1+6x_2)$$

Which is the required linear transformation.

---

3) Verify the Rank-nullity theorem for the $T: R^3 \to R^3$ defined by $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$

Solⁿ:- To find the basis of nullity of T

Let $v=(x,y,z) \in R^3$ such that

Nullity of $T = \{v \in V \mid T(v)=0\}$

$$T(x,y,z) = 0.$$

$$(x+2y-z,\ y+z,\ x+y-2z) = (0,0,0).$$

$$\Rightarrow \quad x+2y-z=0 \Rightarrow x = 3z$$
$$y+z=0 \Rightarrow y=-z.$$
$$x+y-2z=0$$

On solving above equations we get

$$x=3z, \quad y=-z$$

$$\therefore \quad \{(x,y,z)\} = \{3z,-z,z\} = (z\{3,-1,1\})$$

Thus $\{(3,-1,1)\}$ is a basis of nullity of T

& Nullity $(T)=1$.

To find the basis of Range of T.

As $\{(1,0,0),(0,1,0),(0,0,1)\}$ generates R^3

$\Rightarrow \{T(1,0,0),\ T(0,1,0),\ T(0,0,1)\}$ generates range of T

$\to \{(1,0,1),\ (2,1,1),\ (-1,1,-2)\}$ generates range

To find the basis of range,
Consider a matrix.

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + R_2$

$R_3 \to R_3 - R_2$

Thus $\{(1,0,1), (0,1,-1)\}$ form a basis of Range of T & $\dim(\text{Range of } T) = 2$.

$\therefore$ Rank (T) + Nullity (T)

$= 2 + 1$

$= 3 = \dim(R^3)$.

Hence rank-Nullity theorem verified.

---

4) Verify the Rank-Nullity theorem for the $T: R^3 \to R^3$ defined by $T(x,y,z) = (x+2y, y-z, x+2z)$.

Inner Product Spaces and Orthogonality.

Inner Product Space :—

Let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is assigned a real number, denoted by $\langle u, v \rangle$. This function is called a (real) inner product on V if it satisfies the following axioms:

(i) Linear Property : $\langle au_1 + bu_2, v \rangle = a\langle u_1, v\rangle + b\langle u_2, v \rangle$

(ii) Symmetric Property: $\langle u, v \rangle = \langle v, u \rangle$

(iii) Positive Definite Property : $\langle u, u \rangle \geq 0$; and $\langle u, u \rangle = 0$ if and only if $u = 0$.

Norm of a vector :

An inner product, $\langle u, u \rangle$ is nonnegative for any vector u.

$$\|u\| = \sqrt{\langle u, u \rangle} \quad \text{or} \quad \|u\|^2 = \langle u, u \rangle$$

This non negative number is called the norm or length of u.

* Every non zero vector v in V can be multiplied by the reciprocal of its length to obtain the unit vector $\hat{v} = \frac{1}{\|v\|} v$. which is a positive multiple of v.

This process is called normalizing v.

Orthogonality :-

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if

$$\langle u, v \rangle = 0.$$

Problems :-

1) Consider vectors $u = (1,2,4)$, $v = (2,-3,5)$, $w = (4,2,-3)$ in R^3. Find a) $\langle u \cdot v \rangle$ b) $\langle u \cdot w \rangle$ c) $\langle v \cdot w \rangle$ d) $\langle (u+v) \cdot w \rangle$

e) $\|u\|$ f) $\|v\|$

a) $\langle u \cdot v \rangle = 2 - 6 + 20 = 16$

b) $\langle u \cdot w \rangle = 4 + 4 - 12 = -4$

c) $\langle v \cdot w \rangle = 8 - 6 - 15 = -13$.

d) $\langle (u+v) \cdot w \rangle = (3,-1,9) \cdot (4,2,-3) = 12 - 2 - 27 = -17$.

e) $\|u\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1+4+16} = \sqrt{21}$.

f) $\|v\| = \sqrt{4+9+25} = \sqrt{38}$.

---

2) Consider the vectors $u = (1,5)$ and $v = (3,4)$ in R^2.

Find : a) $\langle u, v \rangle$ with respect to the usual inner product in R^2.

b) $\|v\|$ using the inner product in R^2.

3) Consider the following polynomials in $P(t)$ and inner product:

$f(t) = t+2$, $g(t) = 3t-2$, $h(t) = t^2-2t-3$ and

$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$.

(a) find $\langle f, g\rangle$ and $\langle f, h\rangle$ (b) find $\|f\|$ and $\|g\|$

(c) normalize f and g.

Sol^n:- (a) $\langle f, g\rangle = \int_0^1 (t+2)(3t-2)\,dt = \int_0^1 (3t^2+4t-4)\,dt$

$$\langle f, g\rangle = t^3+2t^2-4t\Big|_0^1 = -1.$$

$$\langle f, h\rangle = \int_0^1 (t+2)(t^2-2t-3)\,dt = \frac{t^4}{4} - \frac{7t^2}{2} - 6t\Big]_0^1 = \frac{-37}{4}$$

(b) $\langle f, f\rangle = \int_0^1 (t+2)(t+2)\,dt = \frac{19}{3}$; $\|f\| = \frac{\sqrt{19}}{\sqrt{3}} = \frac{\sqrt{57}}{3}$.

$\langle g, g\rangle = \int_0^1 (3t-2)(3t-2)\,dt = 1$; $\|g\| = \sqrt{1} = 1$

(c) Since $\|f\| = \frac{\sqrt{57}}{3}$ and g is already a unit vector,

$$\hat{f} = \frac{1}{\|f\|} f = \frac{3}{\sqrt{57}}(t+2)$$

$$\hat{g} = \frac{1}{\|g\|} g = 3t-2.$$

---

4) Let $M = M_{2,3}$ with inner product. $\langle A, B\rangle = tr\langle B^T A\rangle$

and let $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 3 & -5 & 2 \\ 1 & 0 & -4 \end{bmatrix}$

Find $\langle A, B\rangle$, $\langle A, C\rangle$, $\langle B, C\rangle$

(a) $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$

$\langle A, B \rangle = 9+16+21+24+25+24 = 119$

$\langle A, C \rangle = 27-40+14+6+0-16 = -9$

$\langle B, C \rangle = 3-10+6+4+0-24 = -21$

(b) $2A+3B = \begin{bmatrix} 21 & 22 & 23 \\ 24 & 25 & 26 \end{bmatrix}$ $4C = \begin{bmatrix} 12 & -20 & 8 \\ 4 & 0 & -16 \end{bmatrix}$

$\langle 2A+3B, 4C \rangle = 252-440+96+0-416 = -324$

(c) $\|A\|^2 = \langle A, A \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$, the sum of the squares of all the elements of A.

$\|A\|^2 = \langle A, A \rangle = 9^2+8^2+7^2+6^2+5^2+4^2 = 271 \Rightarrow \|A\| = \sqrt{271}$

$\|B\|^2 = \langle B, B \rangle = 1^2+2^2+3^2+4^2+5^2+6^2 = 91 \Rightarrow \|B\| = \sqrt{91}$

---

4) Verify the vectors $u=(1,1,1)$, $v=(1,2,-3)$ & $w=(1,-4,3)$ in R^3 are orthogonal or not.

Soln:- $\langle u, v \rangle = 1+2-3 = 0$, $\langle u, w \rangle = 1-4+3 = 0$, $\langle v, w \rangle = 1-8-9 = -16$

Thus u is orthogonal to v and w,

v & w are not orthogonal.