

# Module -2

## Time Response Specifications

**Faculty Handling: -**

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# Contents

- Time Response of feedback control systems
- Standard test signals
- Unit step response of First and Second order Systems.
- Time response specifications
- Time response specifications of second order systems
- steady state errors and error constants.
- Introduction to PI, PD and PID Controllers (excluding design).

# Lesson Plan

S.N	DATE PLANNED	TOPIC	DATE ENGAGED	REMARKS
1.1		Time Response of feedback control systems		
1.2		Standard test signals		
1.3		Unit step response of First order Systems		
1.4		Unit step response of Second order Systems		
1.5		Time response specifications		
1.6		Time response specifications of second order systems		
1.7		steady state errors and error constants		
1.8		Introduction to PI, PD(excluding design).		
1.9		PID Controllers (excluding design).		

# Motivation

- How to know the performance of a control system for any input signal?
- How to design a control system which meets the desired response and control requirements ?

**Time Domain Analysis and Design Specifications**

# Time Domain Analysis

- Time domain analysis refers to the analysis of system performance in time i.e., the study of evolution of system variables (specifically output) with time
- There are two common ways of analyzing the response of systems:
  1. Natural response and forced response
  2. Transient response and steady state response
- In both cases, the complete response of the system is given by the combination of both responses i.e., natural and forced responses or transient and steady state responses

# Natural and Forced Responses

- **Natural response (Zero input response) :**

- System's response to initial conditions with all external forces set to zero

- E.g. In RLC circuits, this would be the response of the circuit to initial conditions (inductor currents or capacitor voltages) with all the independent voltage and current sources set to zero

- **Forced response (Zero state response) :**

- System's response to external forces with zero initial conditions

- E.g. In RLC circuits, this would be the response of the circuit to only external voltage and current source, and zero initial conditions

# Transient and Steady State Responses

## Transient response $\mathbf{y}_{tr}(t)$ :

- –Part of the time response that goes to zero as time tends to be large
- –Transient response can be tied to any event that affects the equilibrium of a system viz. switching, disturbance, change in input, etc.

$$\lim_{t \rightarrow \infty} \mathbf{y}_{tr}(t) = 0$$

## Steady state response $\mathbf{y}_{ss}(t)$ :

- –Steady state response is the time response of a system after transient practically vanishes and as time goes to infinity
- $\mathbf{y}(t) = \mathbf{y}_t(t) + \mathbf{y}_{ss}(t)$

# Standard Test Inputs

- In most cases, the input signals to a control system are not known prior to design of control system
- Hence to analyse the performance of a control system, it is excited with standard test signals
- In general, control system design specifications are also based on the response of the system to such test signals
- **Standard test signals include:**
  - Unit impulse, unit step (sudden change), ramp (constant velocity), parabolic (constant acceleration) and sinusoidal
  - These inputs are chosen because they capture many of the possible variations that can occur in an arbitrary input signal

# Review: Standard Test Inputs

- **Unit impulse signal:**

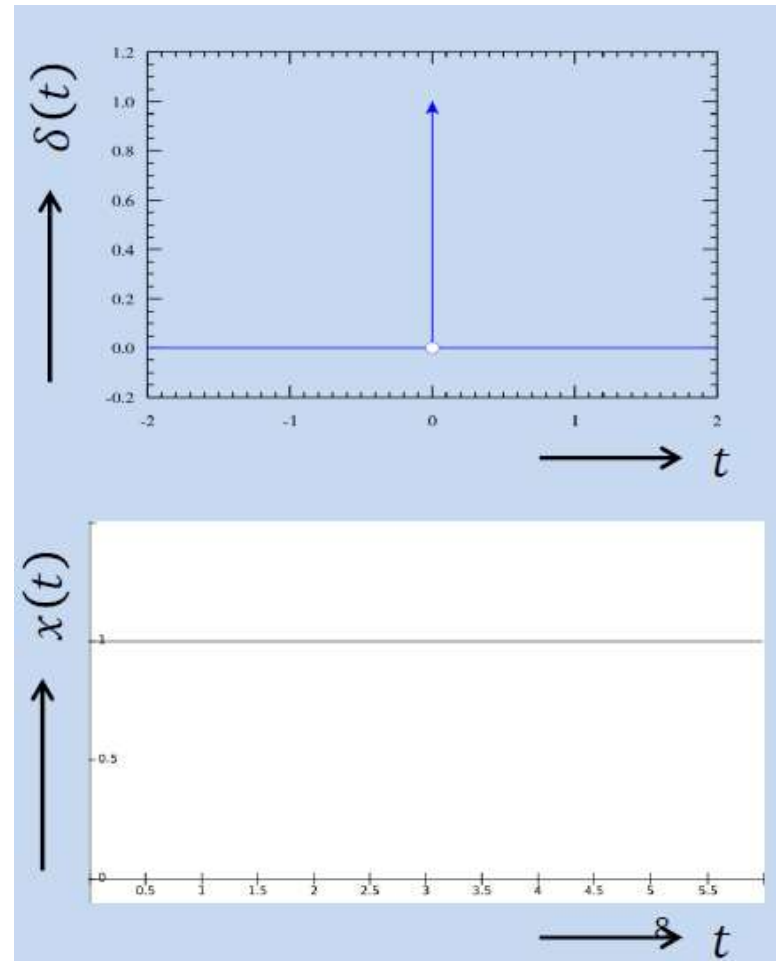
- A signal which is non-zero only at  $t = 0$  and integrates to one

$$\int_{-\infty}^{\infty} \delta(t) = 1$$
$$\mathcal{L}\{\delta(t)\} = 1$$

- **Unit step signal:**

- A signal that switches to one at a time instant and stays there indefinitely

$$x(t) = \begin{cases} 1 & \forall t > 0 \\ 0 & \forall t < 0 \end{cases}$$
$$\mathcal{L}\{x(t)\} = \frac{1}{s}$$



# Review: Standard Test Inputs

- **Ramp signal:**

- A signal which increases linearly with time

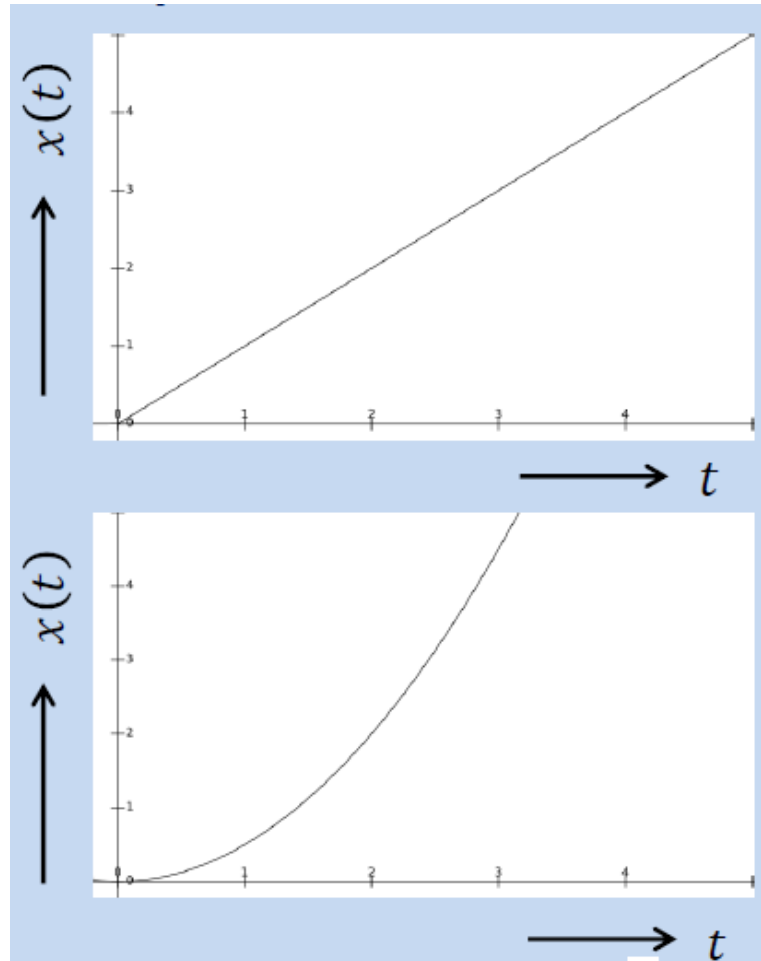
$$x(t) = \begin{cases} At & \forall t \geq 0 \\ 0 & \forall t < 0 \end{cases}$$

$$\mathcal{L}\{x(t)\} = \frac{A}{s^2}$$

- **Parabolic signal:**

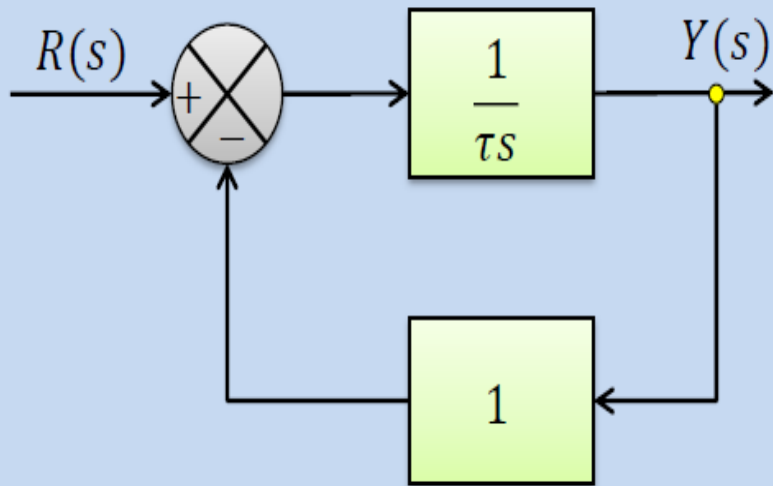
$$x(t) = \begin{cases} \frac{At^2}{2} & \forall t \geq 0 \\ 0 & \forall t < 0 \end{cases}$$

$$\mathcal{L}\{x(t)\} = \frac{A}{s^3}$$



# 1<sup>st</sup> Order Systems

- Systems with only one pole are called 1<sup>st</sup> order systems



Standard block diagram of a 1<sup>st</sup> order system

$$TF = \frac{Y(s)}{R(s)} = \frac{1}{\tau s + 1}$$

$\tau$ : **System time constant**

- It characterizes the speed of response of a system to an input
- Higher the time constant, slower the response and vice-versa

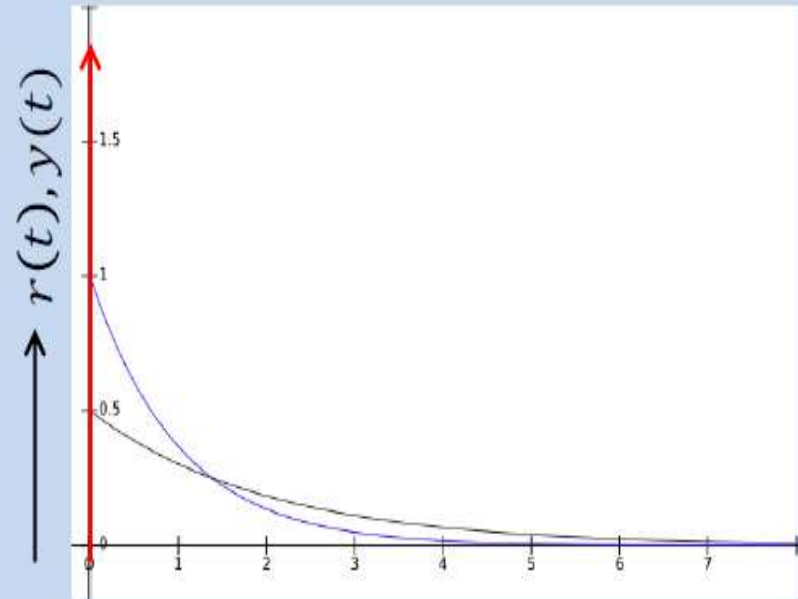
# Impulse Response of 1<sup>st</sup> Order Systems

Unit impulse:  $R(s) = 1$

$$Y(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{\tau s + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$

- $\tau$  is the time constant of the system
- $\frac{1}{\tau} e^{-\frac{t}{\tau}}$  is the transient term  $y_{tr}(t)$  while the steady state term  $y_{ss}(t) = 0$



$r(t)$  and  $y(t)$  when  $\tau = 1, \tau = 2$

# Step Response of 1<sup>st</sup> Order Systems

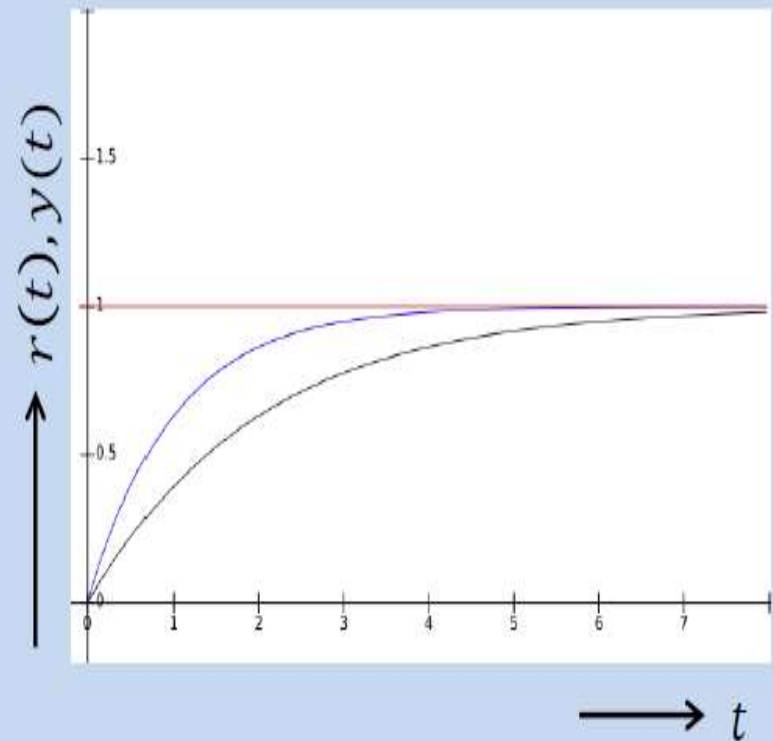
$$\text{Unit step: } R(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{s(\tau s + 1)}$$

$$= \frac{1}{s} - \frac{\tau}{\tau s + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1 - e^{-\frac{t}{\tau}}$$

- In this case,  $t_{tr}(t) = -e^{-\frac{t}{\tau}}$  and  $t_{ss}(t) = 1$



$r(t)$  and  $y(t)$  when  $\tau = 1, \tau = 2$

# Ramp Response of 1<sup>st</sup> Order Systems

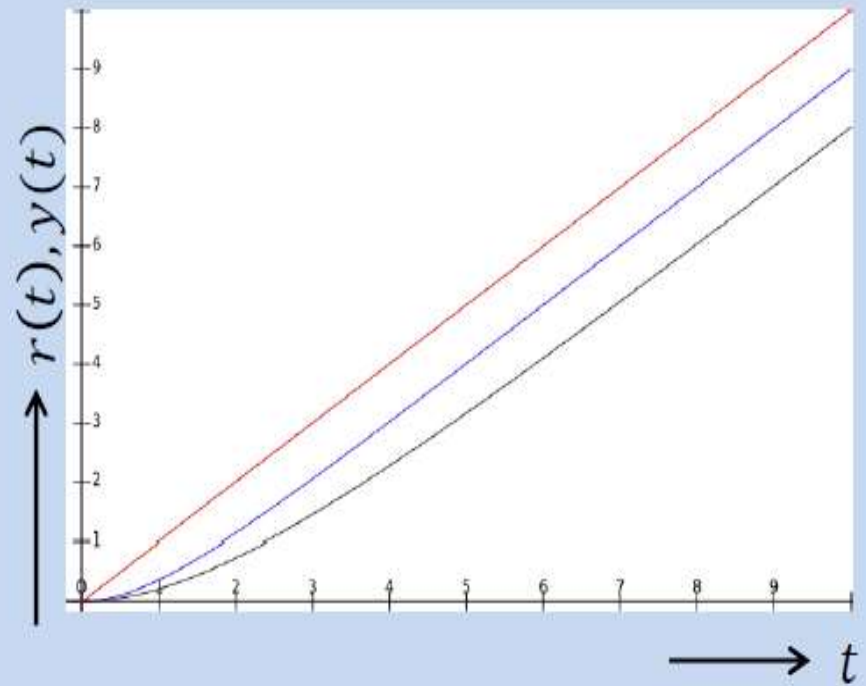
$$\text{Unit ramp: } r(t) = t \Rightarrow R(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{s^2(\tau s + 1)}$$

$$= \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = t - \tau + \tau e^{-\frac{t}{\tau}}$$

$$t_{tr}(t) = \tau e^{-\frac{t}{\tau}} ; t_{ss}(t) = t - \tau$$



$r(t)$  and  $y(t)$  when  $\tau = 1, \tau = 2$

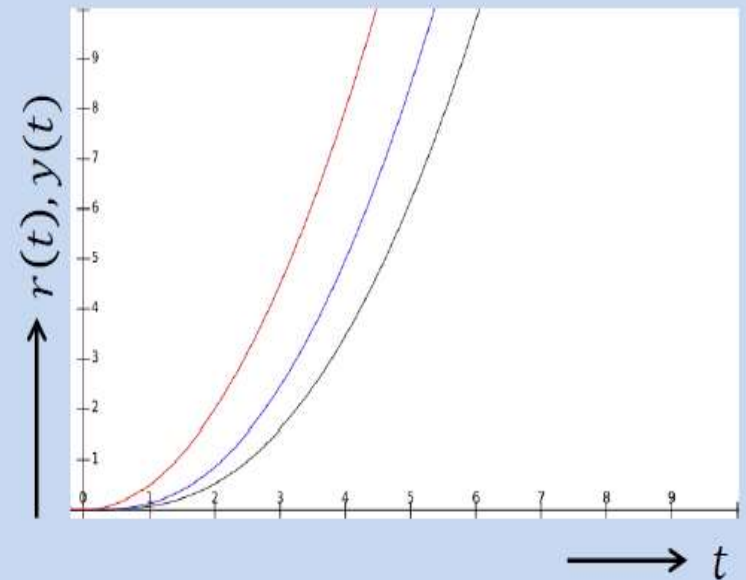
# Parabolic Response of 1<sup>st</sup> Order Systems

$$\text{Unit parabolic: } r(t) = \frac{t^2}{2} \Rightarrow R(s) = \frac{1}{s^3}$$

$$Y(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{s^3(\tau s + 1)}$$
$$= \frac{1}{s^3} - \frac{\tau}{s^2} + \frac{\tau^2}{s} - \frac{\tau^3}{\tau s + 1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{t^2}{2} - \tau t + \tau^2 - \tau^2 e^{-\frac{t}{\tau}}$$

$$t_{tr}(t) = -\tau^2 e^{-\frac{t}{\tau}}; t_{ss}(t) = \frac{t^2}{2} - \tau t + \tau^2$$

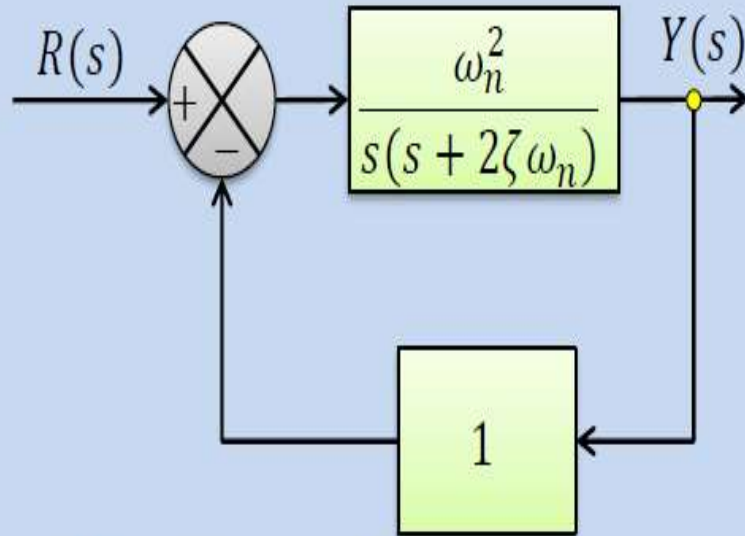


$r(t)$  and  $y(t)$  when  $\tau = 1, \tau = 2$

# 2<sup>nd</sup> Order Systems

- Systems with two poles are called 2<sup>nd</sup> order systems
- E.g. An RLC circuit or mass-spring-damper system
- For an RLC circuit :  $TF = \frac{1}{s^2LC + sRC + 1}$
- For an MSD system :  $TF = \frac{1}{Ms^2 + Bs + K}$
- In general, the transfer function of a 2<sup>nd</sup> order system can be written as:  $TF = \frac{b}{s^2 + as + b}$
- To study and understand the response of a 2<sup>nd</sup> order system, its transfer function is written in terms of certain system parameters

# Standard Form of 2<sup>nd</sup> Order System



Block diagram of a 2<sup>nd</sup> order system

$$TF = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Standard form of transfer function

$\omega_n$ : System natural frequency  
 $\zeta$ : System damping ratio

# Important System Parameters

- **System damping ratio  $\zeta$** : a dimensionless quantity describing the decay of oscillations during a transient response
- *It is the ratio of actual damping to critical damping of a system*
- **System natural frequency  $\omega_n$** : angular frequency at which system tends to oscillate in the absence of damping force
- **System damped frequency  $\omega_d$** : angular frequency at which system tends to oscillate in the presence of damping force

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

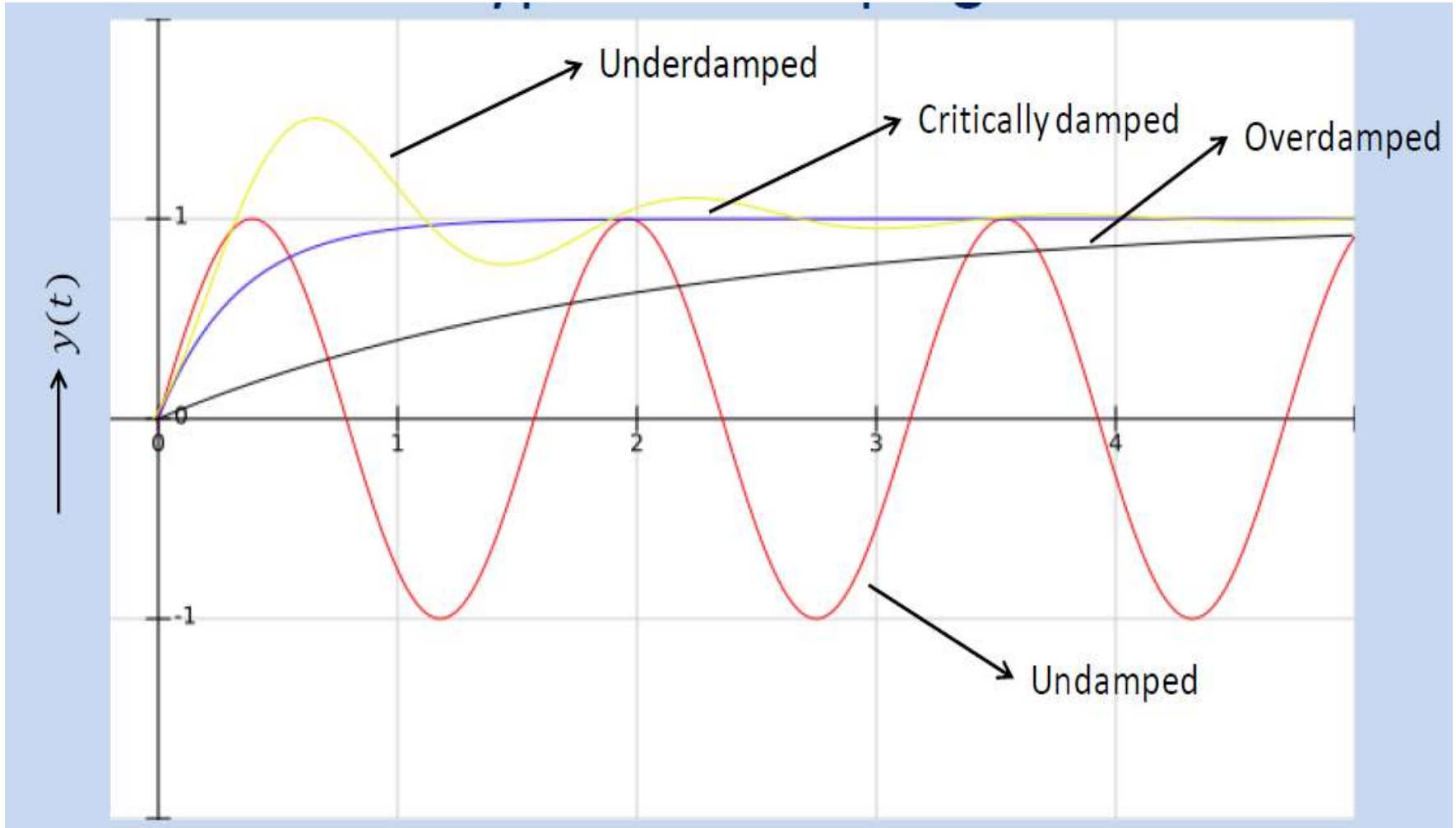
# Response of 2<sup>nd</sup> Order Systems

- Response of 2<sup>nd</sup> order systems mainly depends on the damping ratio  $\zeta$
- For any test input, the response of a 2<sup>nd</sup> order system can be studied in four cases depending on the damping effect created by value of  $\zeta$ :
  1.  $\zeta > 1$ : Overdamped system
  2.  $\zeta = 1$ : Critically damped system
  3.  $0 < \zeta < 1$ : Underdamped system
  4.  $\zeta = 0$ : Undamped system
- **Note:** We do not consider negative damping ratio  $\zeta$  because negative damping actually means the oscillations are increasing in amplitude which results in unstable systems

# Damping and Types of Damping

- Damping is an effect created in an oscillatory system that reduces, restricts or prevents the oscillations in the system
- Systems can be classified as follows depending on damping effect:
  - **Overdamped systems:** Transients in the system exponentially decays to steady state without any oscillations
  - **Critically damped systems:** Transients in the system decay to steady state without any oscillations in shortest possible time
  - **Underdamped systems:** System transients oscillate with the amplitude of oscillation gradually decreasing to zero
  - **Undamped systems:** System keeps on oscillating at its natural frequency without any decay in amplitude

# Types of Damping



# Impulse Response of 2<sup>nd</sup> Order Systems

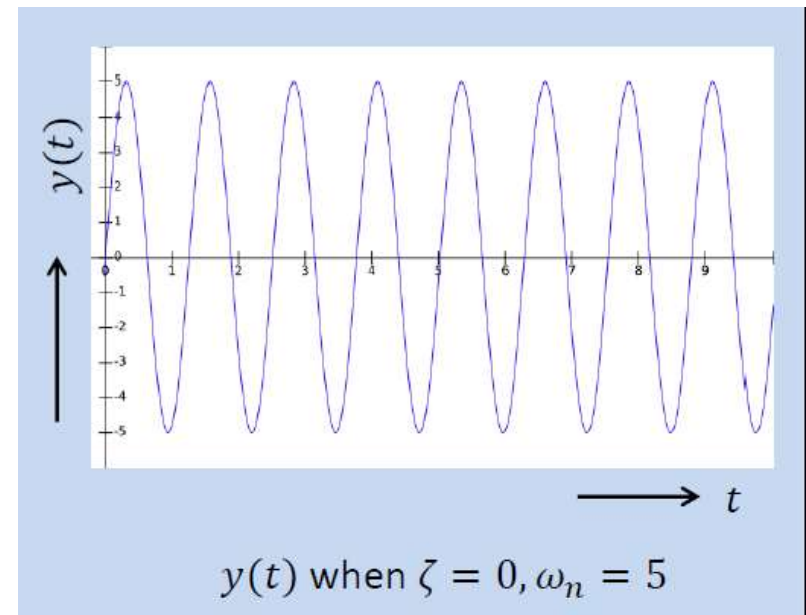
Unit impulse:  $R(s) = 1$

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

**Case 1:**  $\zeta = 0$  – Undamped system

$$Y(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \omega_n \sin(\omega_n t)$$



# Contd..

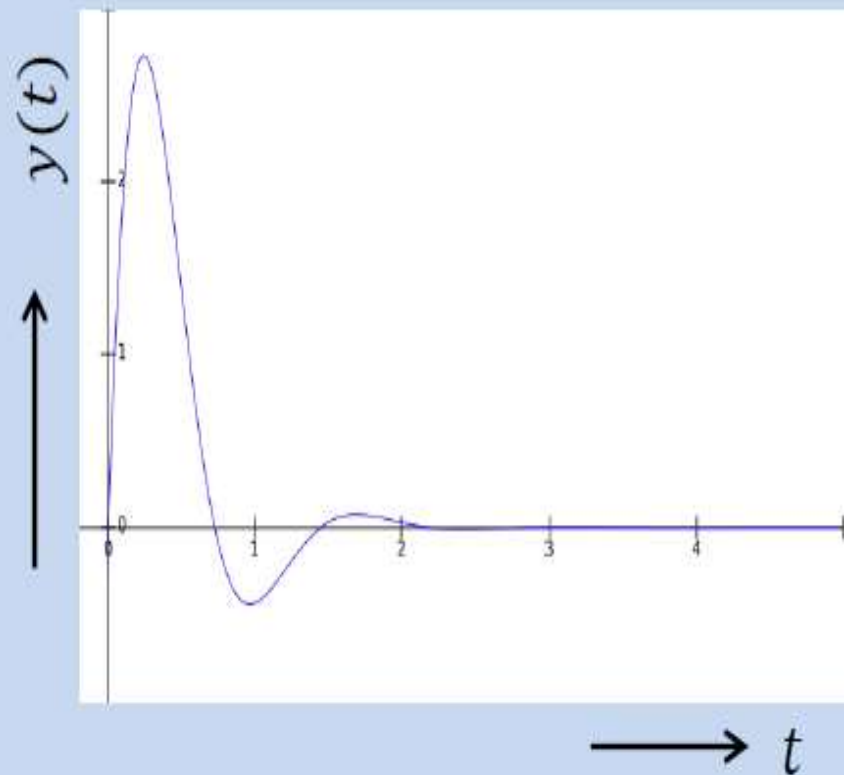
**Case 2:**  $0 < \zeta < 1$  – Underdamped system

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{\omega_n^2}{(s + \zeta\omega_n - j\omega_d)(s + \zeta\omega_n + j\omega_d)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{\left( \frac{\omega_n}{\sqrt{1 - \zeta^2}} \right) \omega_d}{(s + \zeta\omega_n)^2 + (\omega_d^2)} \right\}$$

$$y(t) = \left( \frac{\omega_n}{\sqrt{1 - \zeta^2}} \right) e^{-\zeta\omega_n t} \sin(\omega_d t)$$



$y(t)$  when  $\zeta = 0.5, \omega_n = 5$

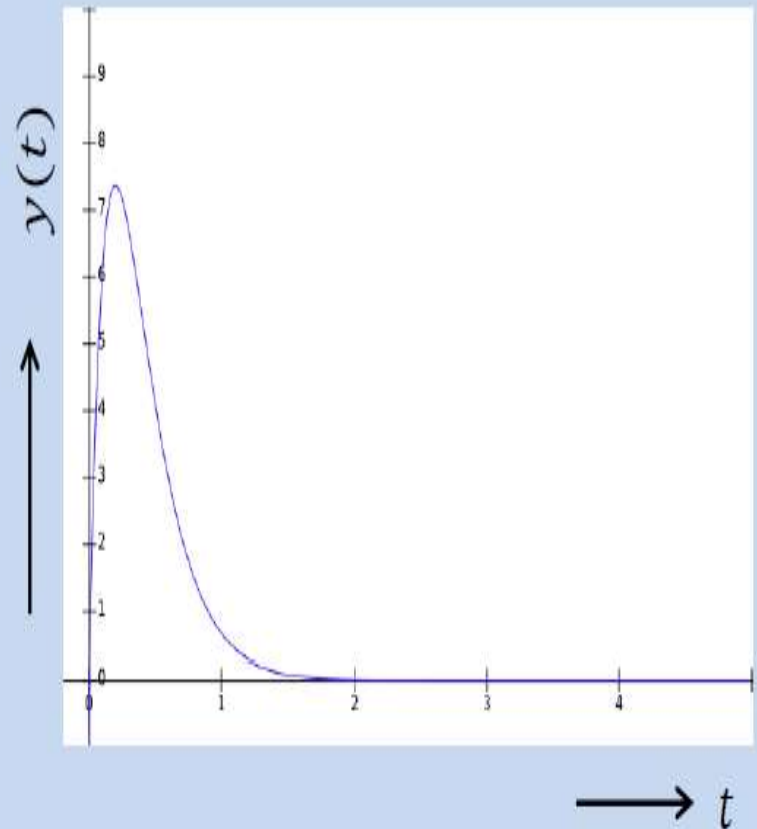
# Contd..

**Case 3:**  $\zeta = 1$  – Critically damped system

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{\omega_n^2}{(s + \omega_n)^2} \right\}$$

$$y(t) = \omega_n^2 t e^{-\omega_n t}$$



$y(t)$  when  $\zeta = 1, \omega_n = 5$

# Contd..

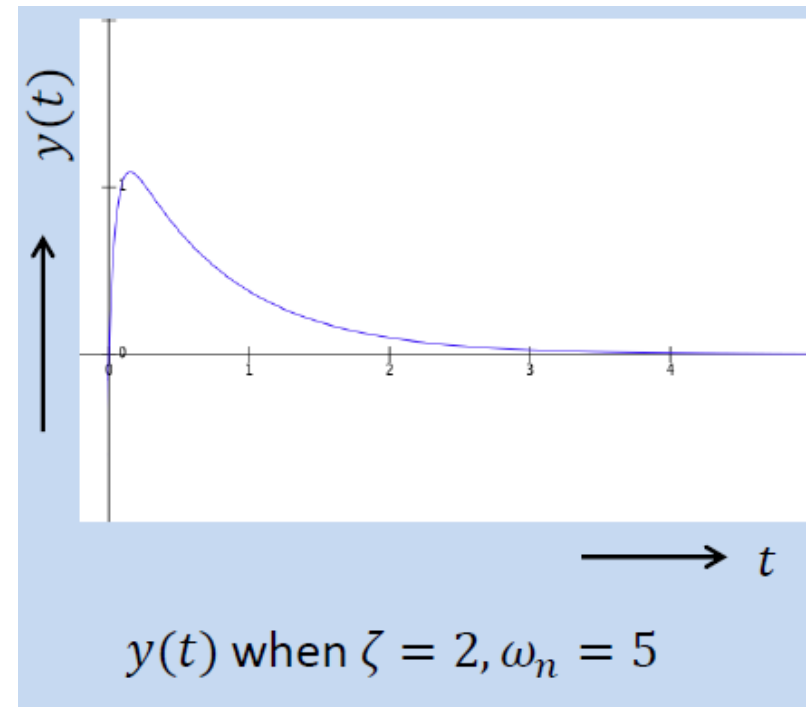
Case 4:  $\zeta > 1$  – Overdamped system

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$y(t)$

$$= \mathcal{L}^{-1} \left\{ \frac{\omega_n^2}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})} \right\}$$

$$y(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-(\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-(\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})t}$$



# Step Response of 2<sup>nd</sup> Order Systems

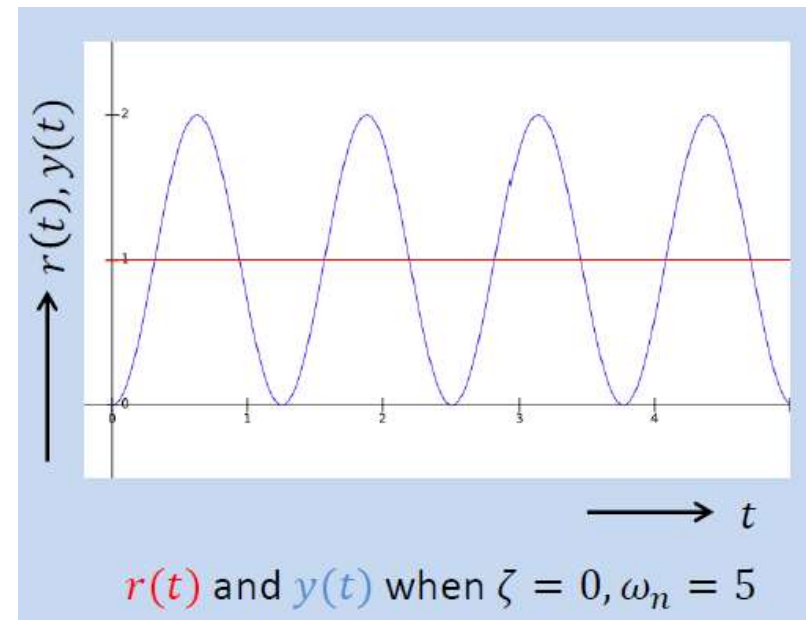
$$\text{Unit step: } R(s) = \frac{1}{s}$$

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

**Case 1:**  $\zeta = 0$  – Undamped system

$$Y(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 1 - \cos(\omega_n t)$$



# Contd..

**Case 2:**  $0 < \zeta < 1$  – Underdamped system

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \Rightarrow y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + (\omega_d^2)} - \frac{\zeta\omega_n \left( \frac{\sqrt{(1-\zeta^2)}}{\sqrt{(1-\zeta^2)}} \right)}{(s + \zeta\omega_n)^2 + (\omega_d^2)} \right\} \rightarrow \omega_d$$

$$y(t) = 1 - e^{-\zeta\omega_n t} \cos(\omega_d t) - \left( \frac{\zeta}{\sqrt{(1-\zeta^2)}} \right) e^{-\zeta\omega_n t} \sin(\omega_d t)$$

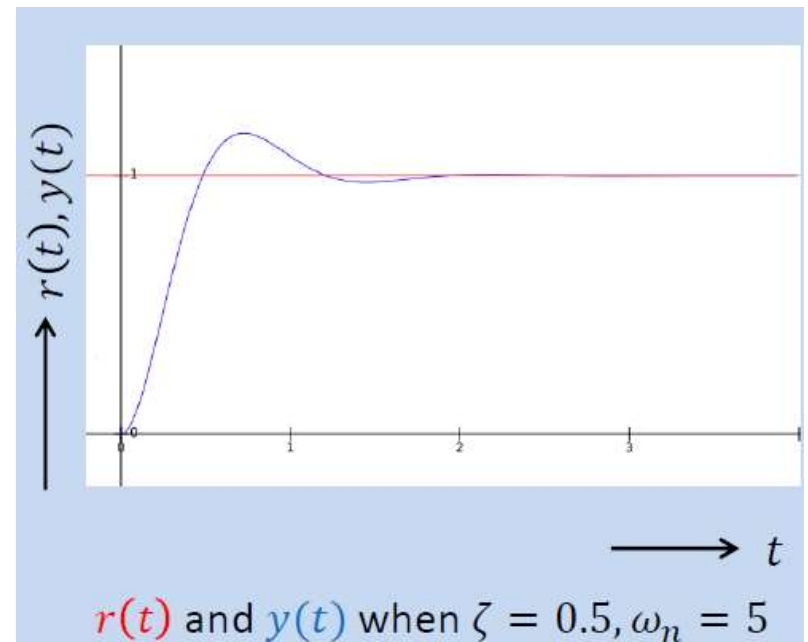
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1-\zeta^2)}} \left[ \sqrt{(1-\zeta^2)} \cos(\omega_d t) + \zeta \sin(\omega_d t) \right]$$

# Contd..

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1 - \zeta^2)}} [\sin\theta \cos(\omega_d t) + \cos\theta \sin(\omega_d t)]$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{(1 - \zeta^2)}} \sin(\omega_d t + \theta)$$

where  $\theta = \cos^{-1} \zeta = \sin^{-1} \sqrt{(1 - \zeta^2)}$



# Contd..

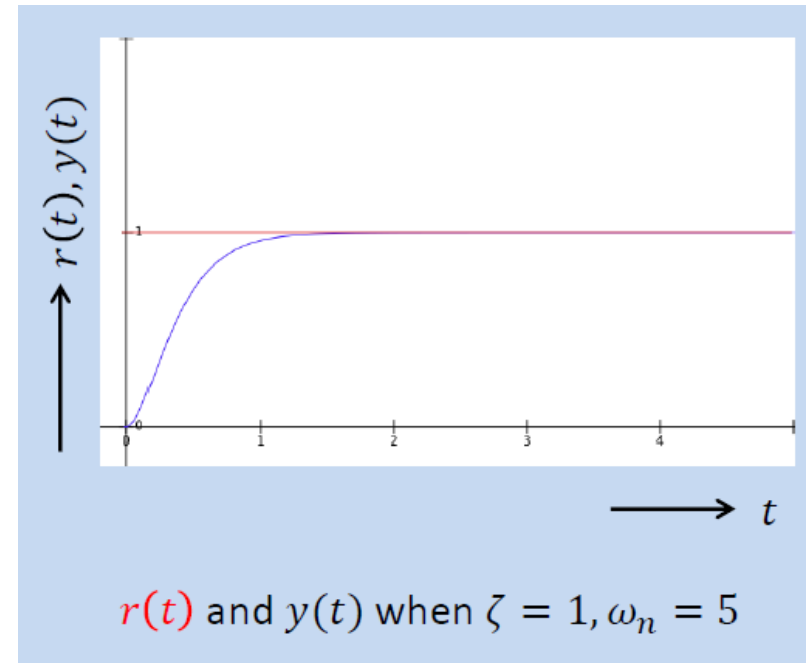
**Case 3:**  $\zeta = 1$  – Critically damped system

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{\omega_n^2}{s(s + \omega_n)^2} \right\}$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2} \right\}$$

$$y(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$



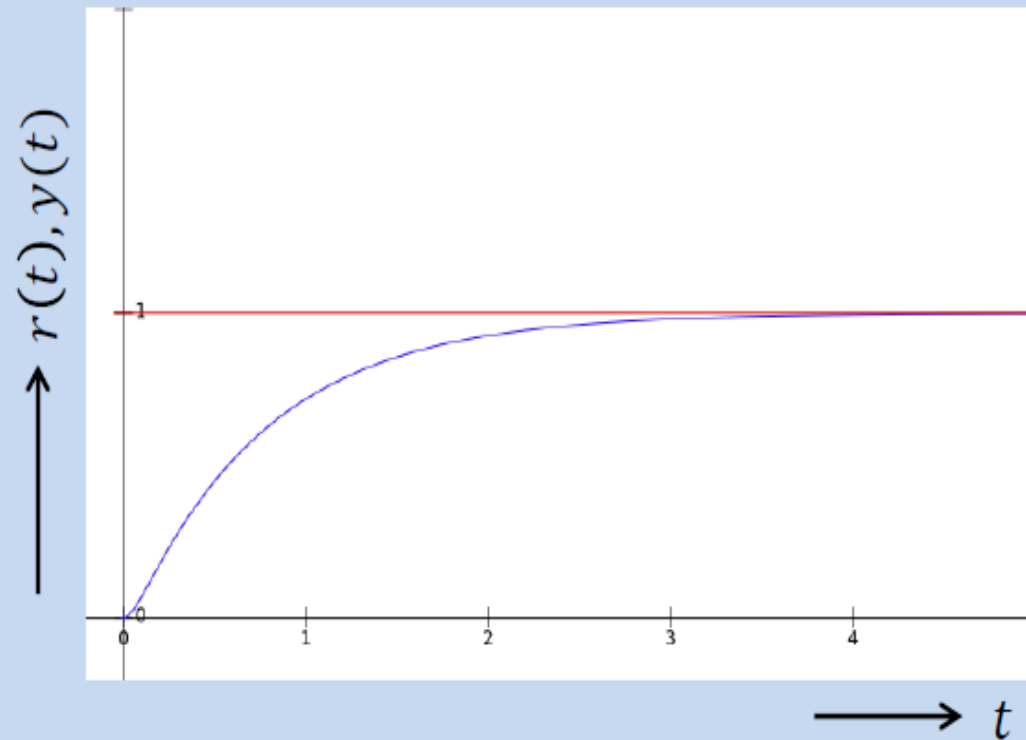
# Contd..

**Case 4:**  $\zeta > 1$  – Overdamped system

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \Rightarrow y(t) = \mathcal{L}^{-1} \left\{ \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \right\}$$
$$= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1 + \frac{\zeta}{\sqrt{(\zeta^2 - 1)}}}{2(s + \zeta\omega_n - \omega_n\sqrt{(\zeta^2 - 1)})} - \frac{\frac{\zeta}{\sqrt{(\zeta^2 - 1)}} - 1}{2(s + \zeta\omega_n + \omega_n\sqrt{(\zeta^2 - 1)})} \right\}$$
$$y(t) = 1 - \frac{1}{2} \left( 1 + \frac{\zeta}{\sqrt{(\zeta^2 - 1)}} \right) e^{-(\zeta\omega_n - \omega_n\sqrt{(\zeta^2 - 1)})t}$$
$$+ \frac{1}{2} \left( \frac{\zeta}{\sqrt{(\zeta^2 - 1)}} - 1 \right) e^{-(\zeta\omega_n + \omega_n\sqrt{(\zeta^2 - 1)})t}$$

# Contd..

Case 4:  $\zeta > 1$  – Overdamped system



$r(t)$  and  $y(t)$  when  $\zeta = 2, \omega_n = 5$

# Time Response Specifications

# Time-domain Specifications: Motivation

- These specifications refer to the performance indices of the step response of a system
- In general, these indices are specified as a part of the design requirements of control systems
- These indices answer the following questions pertaining to step response of a system:
  - How fast the system moves to follow the input ?
  - How oscillatory is the response (indicative of damping) ?
  - How long does it take to practically reach the final value ?

# Time-domain Specifications

**Delay time  $t_d$ :** –Time required for the response to reach 50% of the final value at first instance

$$t_d = (1 + 0.7\zeta) / \omega_n$$

**Rise time  $tr$ :**

–Time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped systems, at first instance

**Peak time  $tp$ :** –Time required for the response to reach the peak value of time response

**Peak overshoot  $M_p$ :** –It is the normalised difference between the peak value of time response and the steady state value

$$M_p = y_{\max} - y_{ss};$$

**Settling time  $ts$ :** –Time required for the response to reach and stay within a specified tolerance band of its final value or steady state value

–Usually the tolerance band is 2% or 5%

Note:  $tp$  and  $M_p$  are not defined for overdamped and critically damped systems

# Time-domain Specifications

1. Steady-state value:  $y_{ss}$

2. Maximum overshoot:

$$\text{Percent overshoot} = \frac{M_p}{y_{ss}} \times 100\%$$

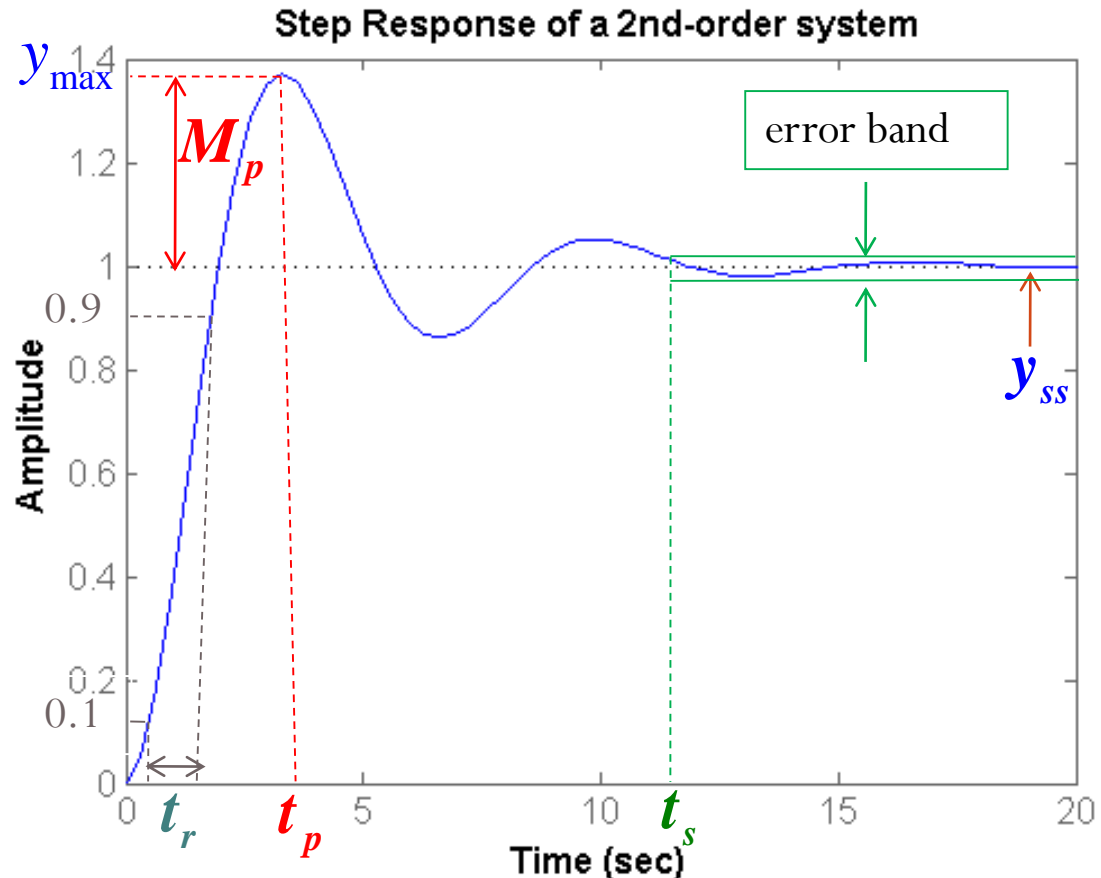
$$M_p = y_{\max} - y_{ss};$$

3. Peak time:  $t_p$

4. Rise time:  $t_r$

5. Settling time:  $t_s$

Is there overshoot in the time response of a first-order system?



# Expression for Rise time, peak time and Overshoot

Unit-step  
response:

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta), \quad 0 < \zeta < 1$$

## 1. Rise Time

$$y(t_r) = 1, \text{ that is, } \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \cdot \sin(\omega_d t_r + \beta) = 0$$
$$\frac{e^{-\zeta \cdot \omega_n \cdot t_r}}{\sqrt{1-\zeta^2}} \neq 0, \Rightarrow \omega_d t_r + \beta = n\pi (n = 0, \pm 1, \pm 2, \dots)$$

$t_r$  is the time needed for the response to reach the steady-state value for the first time, so  $n=1$ .

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \beta}{\omega_n \sqrt{1-\zeta^2}}$$

For a given  $\omega_n$ ,  $\zeta \downarrow$ ,  $t_r \downarrow$ ;  
For a given  $\zeta$ ,  $\omega_n \uparrow$ ,  $t_r \downarrow$ .

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta), \quad 0 < \zeta < 1$$

$$\frac{dy(t)}{dt} = 0$$

## 2. Peak time

$$\frac{dy(t)}{dt} = \left( \frac{\zeta^2 \omega_n}{\sqrt{1-\zeta^2}} + \omega_n \sqrt{1-\zeta^2} \right) e^{-\zeta \cdot \omega_n \cdot t} \sin \omega_d t$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \cdot \omega_n \cdot t} \sin \omega_d t = 0$$

$$\therefore \sin \omega_d t_p = 0 \Rightarrow \omega_d t_p = n\pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

$t_p$  is the time needed for the response to reach the maximum value for the first time, so  $n=1$ .

$$\omega_d t_p = \pi \Rightarrow t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

For a given  $\omega_n$ ,  $\zeta \downarrow$ ,  $t_p \downarrow$ ;  
For a given  $\zeta$ ,  $\omega_n \uparrow$ ,  $t_p \downarrow$ .

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta), \quad 0 < \zeta < 1$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

### 3. Overshoot

$$y(t_p) = 1 - \frac{e^{-\zeta \cdot \omega_n \cdot t_p}}{\sqrt{1-\zeta^2}} \sin(\pi + \beta)$$

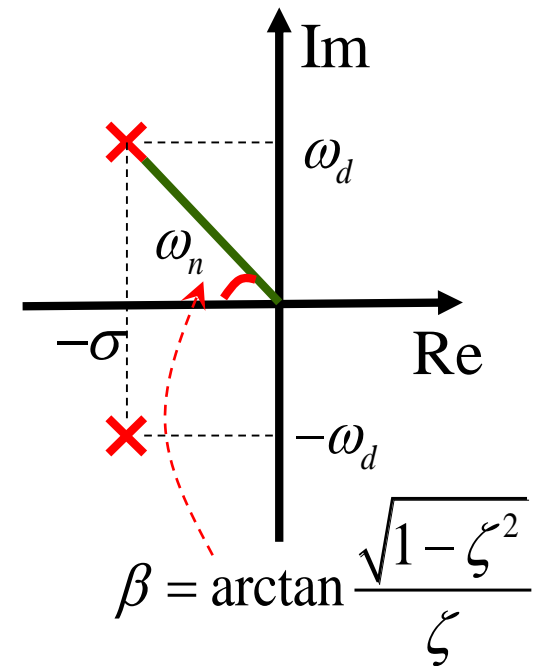
$$\sin(\pi + \beta) = -\sin \beta = -\sqrt{1-\zeta^2}$$

$$\therefore y(t_p) = 1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

Suppose that  $y(\infty) = 1$

$$\text{Thus } \sigma\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \cdot 100\%$$

$$\sigma\% = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$



Overshoot is a function of damping ratio  $\zeta$ , independent of  $\omega_n$ .

# Expression for Settling time

## 4 Settling Time

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta), \quad 0 < \zeta < 1$$

With the definition of error band

$$\left| \frac{e^{-\zeta \cdot \omega_n \cdot t}}{\sqrt{1-\zeta^2}} \cdot \sin(\sqrt{1-\zeta^2} \cdot \omega_n t_s + \beta) \right| \leq 0.05 \text{ or } 0.02$$

$t_s$  can not be obtained directly, but we can get the relationship between  $\omega_n t_s$  and  $\zeta$ .

## Relationship between $(\zeta, \omega_n)$ and $(t_r, t_p, \sigma\%, t_s)$

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \beta}{\omega_n \sqrt{1 - \zeta^2}}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

For a given  $\omega_n$

$$\zeta \downarrow \Rightarrow t_r \downarrow t_p \downarrow$$

$$\zeta \downarrow \Rightarrow \sigma\% \uparrow$$

$$\sigma\% = e^{-\pi\zeta / \sqrt{1 - \zeta^2}} \cdot 100\%$$

$\zeta$  can be calculated by the requirement on the overshoot  $\sigma\%$ .  $\zeta \in [0.4, 0.8]$ .

$$t_s \approx \frac{3}{\zeta\omega_n} (\Delta = 5\%)$$

$$t_s \approx \frac{4}{\zeta\omega_n} (\Delta = 2\%)$$

$$\zeta\omega_n \uparrow \Rightarrow t_s \downarrow$$

Once  $\zeta$  is determined,  $\omega_n$  can be determined based on the requirement on error band  $\Delta\%$ .

# Application of Damped Systems

- **Overdamped systems:**
  - Push button water tap shut-off valves
  - Automatic door closers (can be critically damped also)
- **Critically damped systems:**
  - Elevator mechanism
  - Gun mechanism (returns to neutral position in shortest possible time)
- **Underdamped systems:**
  - All string instruments, bells are underdamped to make sound appealing
  - Analog electrical or mechanical measuring instruments

# Steady State Error

- It is the error between the actual output and the desired output as  $t \rightarrow \infty$

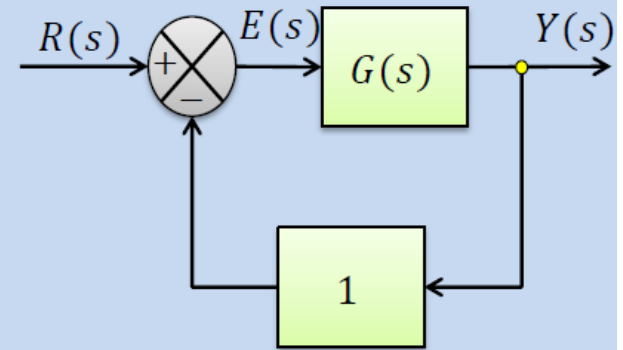
$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} (r(t) - y(t))$$

By final value theorem,

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

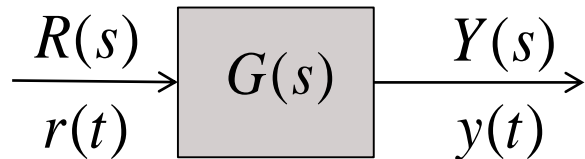
$$E = R - Y = R - \frac{GR}{1 + G} = \frac{R}{1 + G}$$

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

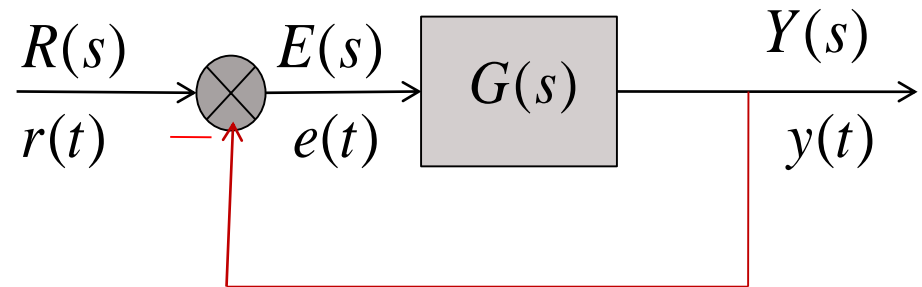


Unity feedback system

## Open-loop control system



## Closed-loop control system



**Error:**  $e(t) = r(t) - y(t)$

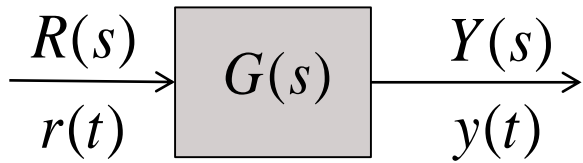
**Steady-state error:**  $e_{ss} = \lim_{t \rightarrow \infty} e(t)$

Utilizing the **final value theorem:**  $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

Assuming  $r(t)=1(t)$  is a unit-step input, according to the above definition, could you calculate the steady-state error of the open-loop and closed-loop control systems?

## Open-loop control system



Unit-step input  $r(t)=1(t)$ ,

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - G(s)R(s) \\ &= [1 - G(s)]R(s) \end{aligned}$$

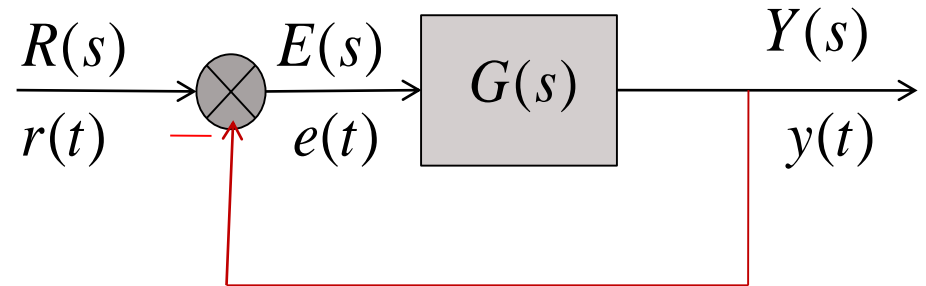
$$e_{ss} = \lim_{s \rightarrow 0} sE(s)$$

$$= \lim_{s \rightarrow 0} s[1 - G(s)] \frac{1}{s}$$

$$= \lim_{s \rightarrow 0} [1 - G(s)]$$

$$= 1 - G(0)$$

## Closed-loop control system



$$R(s) = \frac{1}{s}$$

$$E(s) = R(s) - Y(s)$$

$$= R(s) - \frac{G(s)}{1 + G(s)} R(s)$$

$$= \frac{1}{1 + G(s)} R(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + G(s)} \frac{1}{s}$$

$$= \lim_{s \rightarrow 0} \frac{1}{1 + G(s)} = \frac{1}{1 + G(0)}$$

Feedback has the effect to reduce steady-state error.

The forward-path transfer function  $G(s)$  can be formulated as

$$G(s) = \frac{k(\tau_1 s + 1) \cdots (\tau_2^2 s^2 + 2\xi_1 \tau_2 s + 1)}{s^v (T_1 s + 1) \cdots (T_2^2 s^2 + 2\xi_2 T_2 s + 1)}$$

$$= \frac{k}{s^v} G_0(s) \quad \text{when } s \rightarrow 0, \quad G_0(s) \rightarrow 1$$

$v$  is the order of the pole of  $G(s)$  at  $s=0$

System Type: the order of the pole of  $G(s)$  at  $s=0$ .

When  $v=0,1,2$ , the system is called type 0, type 1, type 2;  $k$  is called open-loop gain.

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + \frac{k}{s^v} G_0(s)} R(s)$$

# Steady State Error for Standard Inputs

- **Unit step input:**  $R(s) = \frac{1}{s}$   
$$e_{ss} = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \rightarrow 0} \frac{1}{1 + G(s)} = \frac{1}{1 + K_p}$$

where  $K_p = \lim_{s \rightarrow 0} G(s)$  is called position error constant

- **Unit ramp (velocity) input:**

$$R(s) = \frac{1}{s^2}$$
$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{s + sG(s)} = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{K_v}$$

where  $K_v = \lim_{s \rightarrow 0} sG(s)$  is called velocity error constant

**Note:** Velocity error is not error in the velocity but it is error in position due to ramp input

- **Unit parabolic (acceleration) input:**

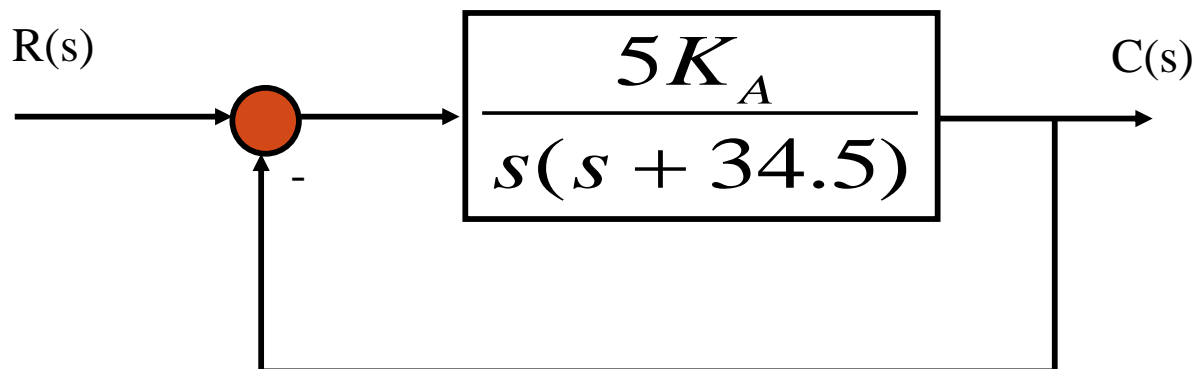
$$R(s) = \frac{1}{s^3}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{s^2 + s^2 G(s)} = \lim_{s \rightarrow 0} \frac{1}{s^2 G(s)} = \frac{1}{K_a}$$

where  $K_a = \lim_{s \rightarrow 0} s^2 G(s)$  is called acceleration error constant

- The error constants  $K_p$ ,  $K_v$  and  $K_a$  describe the ability of a system to reduce or eliminate steady state errors
- These values mostly depend on the type of the system
- As the type of the system becomes higher, more steady-state errors are eliminated

**Example 1:** Consider the following unit-feedback system



System input is the **unit-step function**, When the **amplifier gains** are  $K_A=200$ ,  $K_A=1500$ ,  $K_A=13.5$  respectively, can you calculate the time-domain specifications of the unit-step response ?

Investigate the effect of the amplifier gain  $K_A$  on the system response

**Solution:** The closed-loop transfer function is

$$\phi(s) = \frac{G(s)}{1 + G(s)} = \frac{5K_A}{s^2 + 34.5s + 5K_A}$$

$$K_A = 200, \therefore \phi(s) = \frac{1000}{s^2 + 34.5s + 1000}$$

$$\therefore \omega_n^2 = 1000, \quad 2\zeta\omega_n = 34.5$$

$$\therefore \omega_n = 31.6(\text{rad} / \text{s}), \quad \zeta = \frac{34.5}{2\omega_n} = 0.545$$

According to the formula to calculate the performance indices, it follows that

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = 0.12(\text{sec})$$

$$t_s \approx \frac{3}{\zeta \omega_n} = 0.174(\text{sec})$$

$$\sigma\% = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\% = 13\%$$

$$K_A = 1500$$

If  $K_A = 200$ , then  $\omega_n = 34.5(\text{rad} / \text{s})$ ;  $\zeta = 0.545$

$$\therefore t_p = 0.12(\text{s}), \quad t_s = 0.174(\text{s}), \quad \sigma\% = 13\%$$

If  $K_A = 1500$ , then  $\omega_n = 86.2(\text{rad} / \text{s})$ ;  $\zeta = 0.2$

$$\therefore t_p = 0.037(\text{s}), \quad t_s = 0.174(\text{s}), \quad \sigma\% = 52.7\%$$

Thus, the greater the  $K_A$ , the less the  $\zeta$ , the greater the  $\omega_n$ , the less the  $t_p$ , the greater the  $\sigma\%$ , while **the settling time  $t_s$  has no change.**

$$K_A = 13.5$$

When  $K_A = 13.5$ ,  $\omega_n = 8.22(\text{rad} / \text{s})$ ,  $\zeta = 2.1$

$\zeta > 1$   
Overdamped

When  $K_A = 13.5$ ,  $\omega_n = 8.22(\text{rad} / \text{s})$ ,  $\zeta = 2.1$

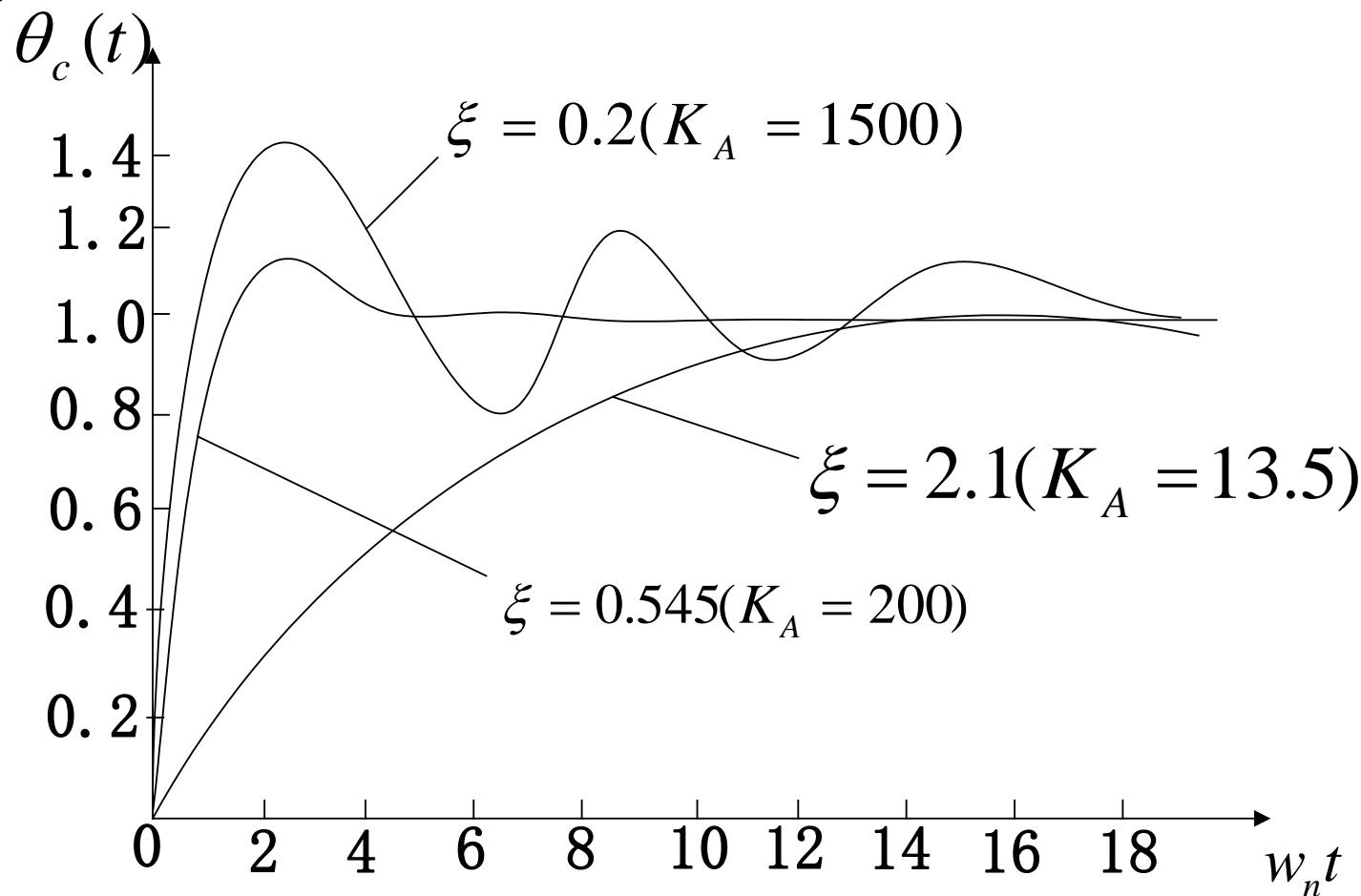
When system is over-damped, there is no peak time, overshoot and oscillation.

The settling time can be calculated approximately:

$$t_s \approx 3T = 1.46(\text{sec})$$

$$\frac{1}{T} = \omega_n (\zeta - \sqrt{\zeta^2 - 1})$$

The settling time is greater than previous cases, although the response has no overshoot, the transition process is very slow, the curves are as follows:



**Note:** When  $K_A$  increases,  $t_p$  decreases,  $t_r$  decreases, the speed of response increases, meanwhile, the overshoot increases. Therefore, to improve the dynamic performance indexes of system, we adopt PD-control or velocity feedback control, namely, PD compensation.

# Features of Steady State Error

- Steady state error is a measure of system accuracy
- In an ideal scenario, the system should match the reference input as time progresses
- It means the steady state error should be as low as possible and hence it is an important performance measure
- Steady state errors depend on two factors:
  1. Type of the reference input  $R(s)$  – step, ramp or parabolic
  2. Type of the system  $G(s)$
- Steady state errors are calculated only for closed loop stable systems

# Type of a System

- Consider following pole-zero form of open loop transfer function of a system:

$$G(s) = \frac{K'((s + z_1)(s + z_2) \dots)}{s^n(s + p_1)(s + p_2) \dots}$$

- Term  $s^n$  in the denominator denotes the number of the poles ( $n$ ) at origin
- System with  $n$  poles at origin is called Type- $n$  system
- $n$  also indicates the number of integrations  $\left(\frac{1}{s}\right)$  in the system
- As  $s \rightarrow 0$ ,  $s^n$  term dominates in determining the steady state error

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + \frac{k}{s^0} G_0(s)} R(s)$$

$v=0$ , type 0 system

Step input:

$$r(t) = 1(t) \quad R(s) = \frac{1}{s}$$

$$e_{ss} = \frac{1}{1 + k}$$

**Steady-state error exists and is finite.**

Ramp input:

$$r(t) = t \quad R(s) = \frac{1}{s^2}$$

$$e_{ss} = \infty$$

**Unstable**

Parabolic input:

$$r(t) = \frac{1}{2}t^2 \quad R(s) = \frac{1}{s^3}$$

$$e_{ss} = \infty$$

**Unstable**

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + \frac{k}{s^1} G_0(s)} R(s)$$

$v=1$ , type 1 system

Step input:

$$r(t) = 1(t) \quad R(s) = \frac{1}{s}$$

$$e_{ss} = \frac{1}{1 + \infty} = 0$$

**No steady-state error**

Ramp input:

$$r(t) = t \quad R(s) = \frac{1}{s^2}$$

$$e_{ss} = \frac{1}{k}$$

**Steady-state error exists**

Parabolic input:

$$r(t) = \frac{1}{2}t^2 \quad R(s) = \frac{1}{s^3}$$

$$e_{ss} = \infty$$

**Unstable**

Type-1 system can track step signal accurately.

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + \frac{k}{s^2} G_0(s)} R(s)$$

$v=2$ , type 2 system

Step input:

$$r(t) = 1(t) \quad R(s) = \frac{1}{s}$$

$$e_{ss} = \frac{1}{1 + \infty} = 0$$

**No steady-state error**

Ramp input:

$$r(t) = t \quad R(s) = \frac{1}{s^2}$$

$$e_{ss} = 0$$

**No steady-state error**

Parabolic input:

$$r(t) = \frac{1}{2}t^2 \quad R(s) = \frac{1}{s^3}$$

$$e_{ss} = \frac{1}{k}$$

**Steady-state error exists**

Type-2 system can track step and ramp signals accurately.

# Steady-state error constants

with **step input**

$$k_p = \lim_{s \rightarrow 0} G(s)$$

- step-error constant

with **ramp input**

$$k_v = \lim_{s \rightarrow 0} sG(s)$$

- ramp-error constant

with **parabolic input**

$$k_a = \lim_{s \rightarrow 0} s^2 G(s)$$

- parabolic-error constant

Type of System	Error constants			Steady-state error $e_{SS}$		
	$k_p$	$k_v$	$k_a$	$r(t) = R_0 \cdot 1(t)$	$r(t) = V_0 t$	$r(t) = A_0 t^2 / 2$
<b>0</b>	$k$	0	0	$\frac{R_0}{1+k}$	$\infty$	$\infty$
<b>I</b>	$\infty$	$k$	0	0	$\frac{V_0}{k}$	$\infty$
<b>II</b>	$\infty$	$\infty$	$k$	0	0	$\frac{A_0}{k}$

 Summary of steady-state error and error constants

for unit-feedback systems ( $H(s)=1$ )

# Steady State Error for Different Systems

- **Type-0 system:**

$$G(s) = \frac{K'((s + z_1)(s + z_2) \dots)}{(s + p_1)(s + p_2) \dots}$$

$$\text{➤ } e_{ss}(\text{position}) = \lim_{s \rightarrow 0} \frac{1}{1+G(s)} = \frac{1}{1+K_p}$$

$$\text{➤ } e_{ss}(\text{velocity}) = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{0} = \infty$$

$$\text{➤ } e_{ss}(\text{acceleration}) = \lim_{s \rightarrow 0} \frac{1}{s^2G(s)} = \frac{1}{0} = \infty$$

- Constant position error, infinite velocity and acceleration errors at steady state

- **Type-1 system:**

$$G(s) = \frac{K'((s + z_1)(s + z_2) \dots)}{s(s + p_1)(s + p_2) \dots}$$

- $e_{ss}(\text{position}) = \lim_{s \rightarrow 0} \frac{1}{1+G(s)} = \frac{1}{1+\infty} = 0$
- $e_{ss}(\text{velocity}) = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{K_v}$
- $e_{ss}(\text{acceleration}) = \lim_{s \rightarrow 0} \frac{1}{s^2G(s)} = \frac{1}{0} = \infty$

- Zero position error, a constant velocity error and infinite acceleration error at steady state

- **Type-2 system:**

$$G(s) = \frac{K'((s + z_1)(s + z_2) \dots)}{s^2(s + p_1)(s + p_2) \dots}$$

- $e_{ss}(\text{position}) = \lim_{s \rightarrow 0} \frac{1}{1+G(s)} = \frac{1}{1+\infty} = 0$

- $e_{ss}(\text{velocity}) = \lim_{s \rightarrow 0} \frac{1}{sG(s)} = \frac{1}{\infty} = 0$

- $e_{ss}(\text{acceleration}) = \lim_{s \rightarrow 0} \frac{1}{s^2G(s)} = \frac{1}{K_a}$

- Zero position error, zero velocity error and a constant acceleration error at steady state

## Example 1

Consider a first-order system with the following parameters.

Time constant  $\tau = 0.1s$  ; DC gain  $K = 20$

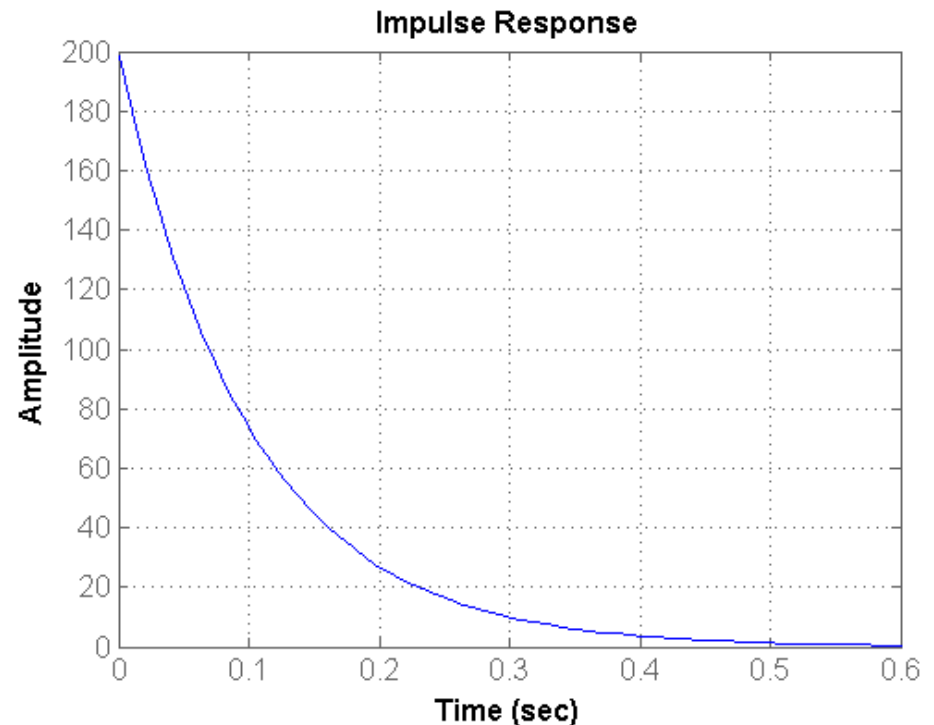
The problem is to determine the unit impulse response of a system that has these parameters.

**Solution.** 
$$y(t) = \frac{K}{\tau} e^{-t/\tau} = \frac{20}{0.1} e^{-t/0.1} = 200e^{-10t}$$

Using MATLAB to get its impulse response

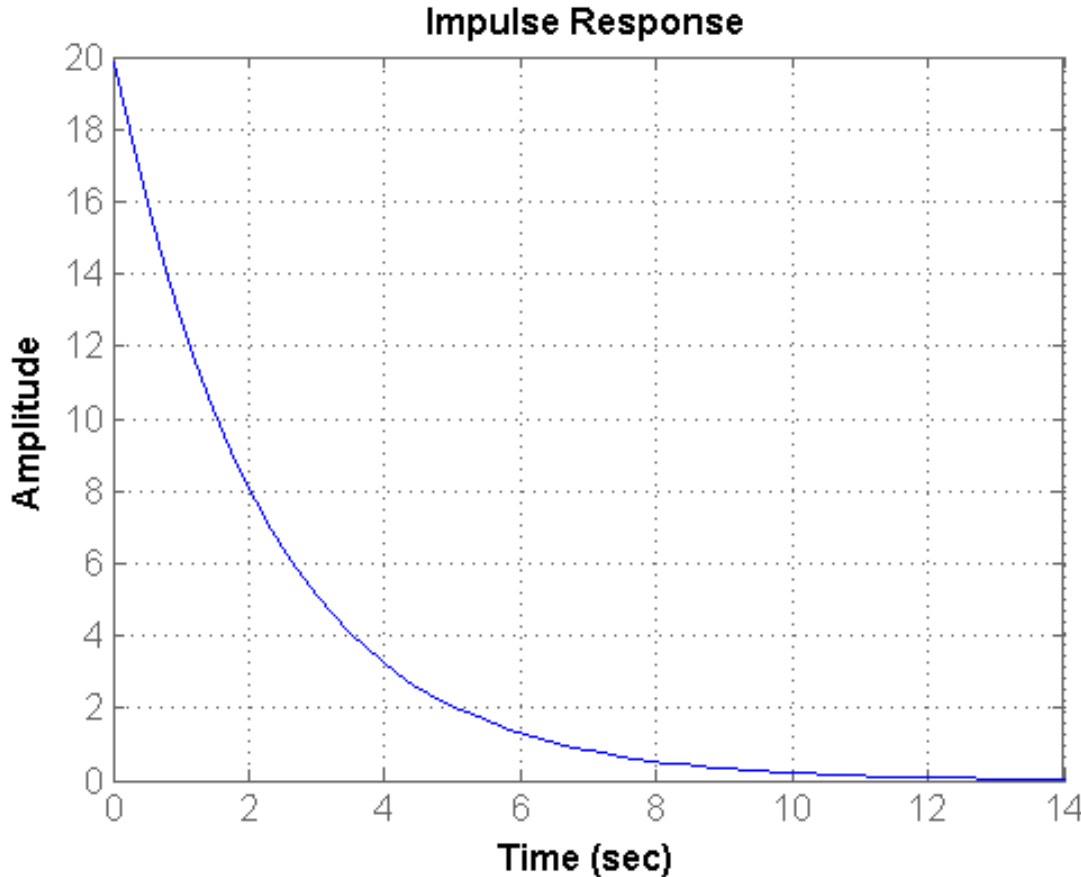
$$G(s) = \frac{K}{\tau s + 1} = \frac{20}{0.1s + 1}$$

```
>> num=20;  
>> den=[0.1 1];  
>> impulse(num,den)
```



## Example 2

Below is the impulse response of a system - i.e. the response to a unit impulse.



The system starts with an initial condition of zero just before the impulse comes along at  $t = 0$ , so  $y(0^-) = 0$ .

## Solution.

The general time response of a first-order system is

$$y(t) = \frac{K}{\tau} e^{-t/\tau} \quad (1)$$

From the right figure, we have

$$\text{at } t = 0, y(0) = 20$$

So we can get

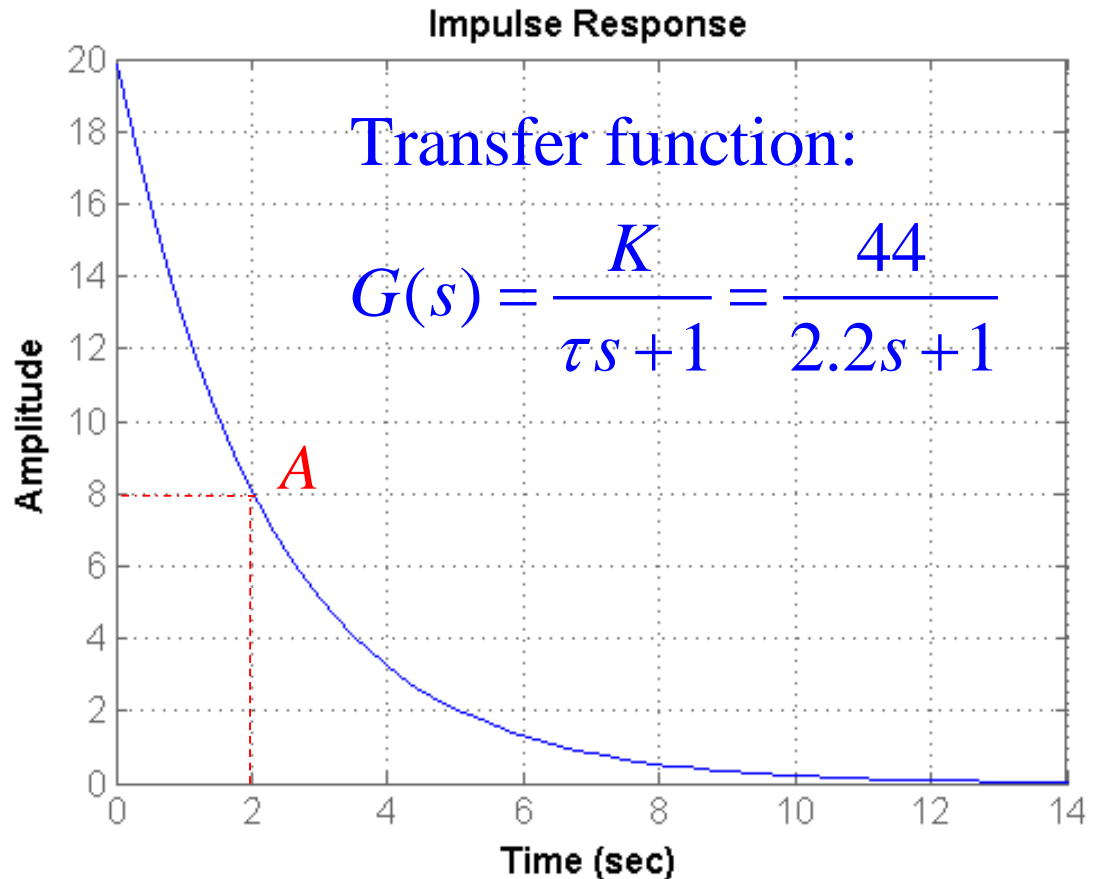
$$\frac{K}{\tau} = 20 \quad (2)$$

How to get  $K$ ?

Pick up a point on the response curve:

$$\mathbf{A:} \quad \text{at } t = 2 \text{ sec}; y(2) = 8$$

$$\mathbf{66} \quad y(2) = \frac{K}{\tau} e^{-2/\tau} = 8 \quad (3)$$



With (2) and (3), we have

$$y(2) = 20e^{-2/\tau} = 8$$

Solving the equation yields:  $e^{-2/\tau} = 0.4$

$$-2 / \tau = \ln(0.4) = -0.9163$$

$$\tau \approx 2.2 \text{ sec} \quad K = 20\tau = 44$$

### Example 3

Consider a first-order system with the following parameters.

Time constant  $\tau = 0.1s$  ; DC gain  $K = 20$

The problem is to determine the unit step response of a system that has these parameters.

**Solution.**

Using the general form of the unit-step response of first –order systems

$$y(t) = K(1 - e^{-t/\tau})$$

With the parameters above, the impulse response is:

$$y(t) = 20(1 - e^{-t/0.1}) = 20(1 - e^{-10t})$$

### Example 3 Time constant $\tau = 0.1s$ ; DC gain $K = 20$

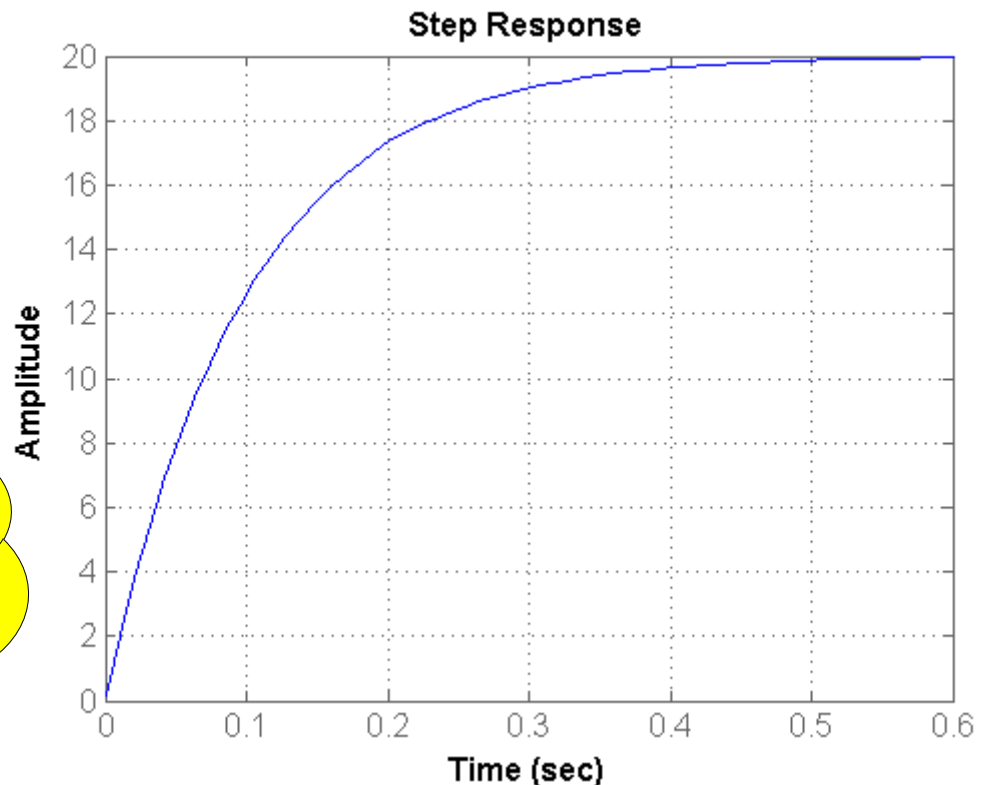
**Solution.** Unit-step time response:  $y(t) = 20(1 - e^{-10t})$

Transfer function of the first-order system:  $G(s) = \frac{K}{\tau s + 1} = \frac{20}{0.1s + 1}$

Using MATLAB to get its step response

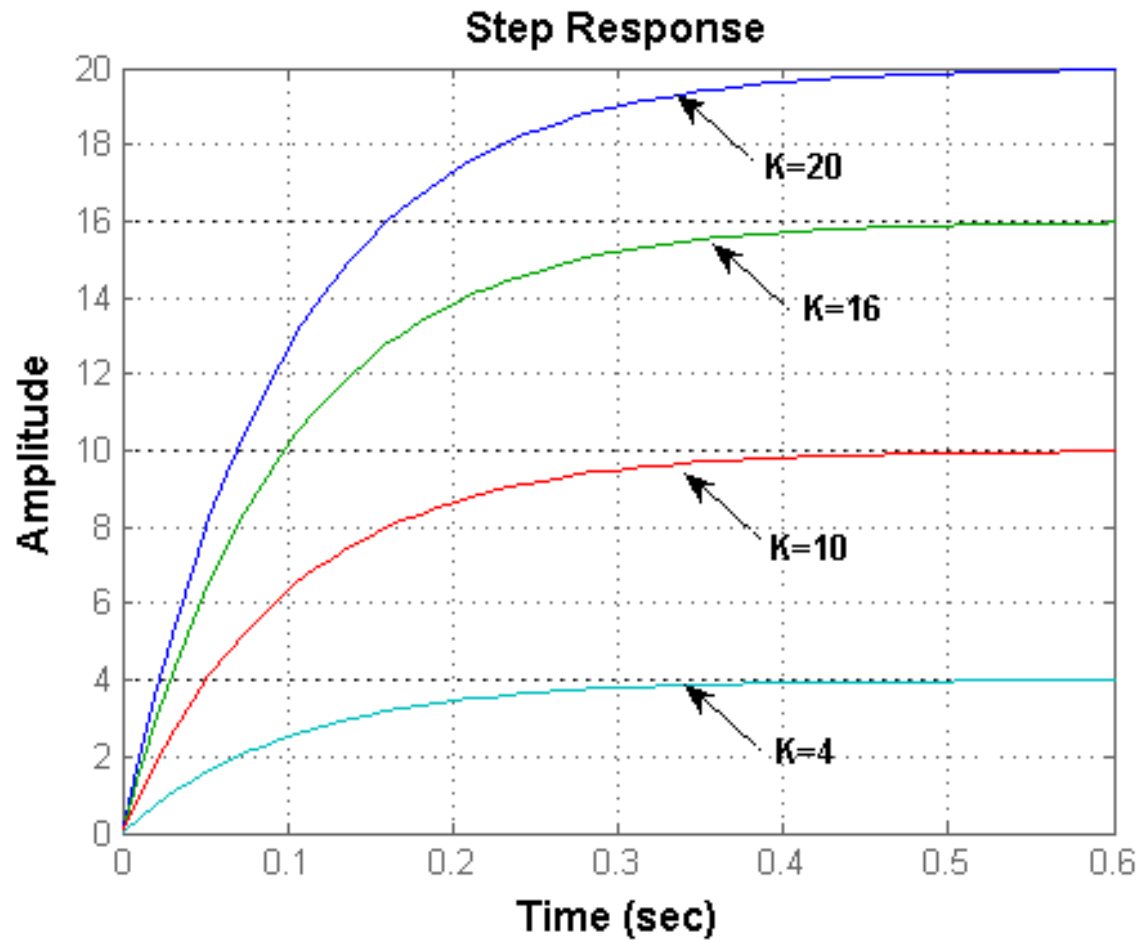
```
>> num=20;  
>> den=[0.1 1];  
>> step(num,den)
```

If we keep  $\tau = 0.1s$  but change  $K$ , what will happen?



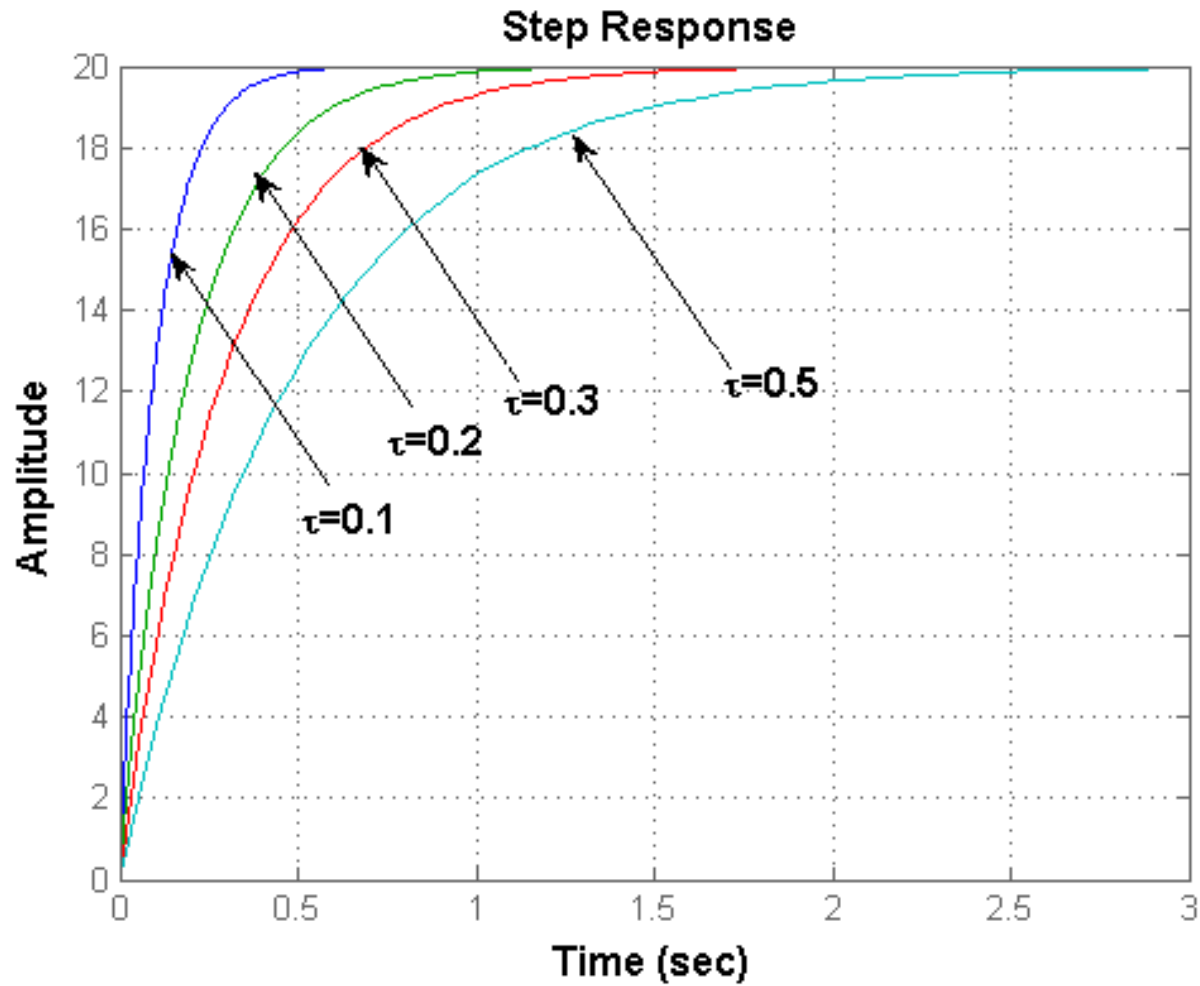
Time constant  $\tau = 0.1s$  ;

DC gain  $K = 20 \rightarrow 16 \rightarrow 10 \rightarrow 4$



DC gain  $K = 20$

Time constant  $\tau = 0.1s \rightarrow 0.2s \rightarrow 0.3s \rightarrow 0.5s$



**Note:** the time constant reflects the system inertia. The smaller is the system inertia, the shorter is  $\tau$  and the quicker is the response, vice versa.

Two important properties of the unit-step response of a first-order system:

a) The time constant  $\tau$  can be used to calculate the system output

$t = \tau$ ,  $y(\tau) = 0.632y(\infty) = 63.3\%$  final value;

$t = 2\tau$ ,  $y(2\tau) = 0.865y(\infty) = 86.5\%$  final value;

$t = 3\tau$ ,  $y(3\tau) = 0.950y(\infty) = 95.0\%$  final value;

$t = 4\tau$ ,  $y(4\tau) = 0.982y(\infty) = 98.2\%$  final value;

experiment methods

-- estimate the time constant

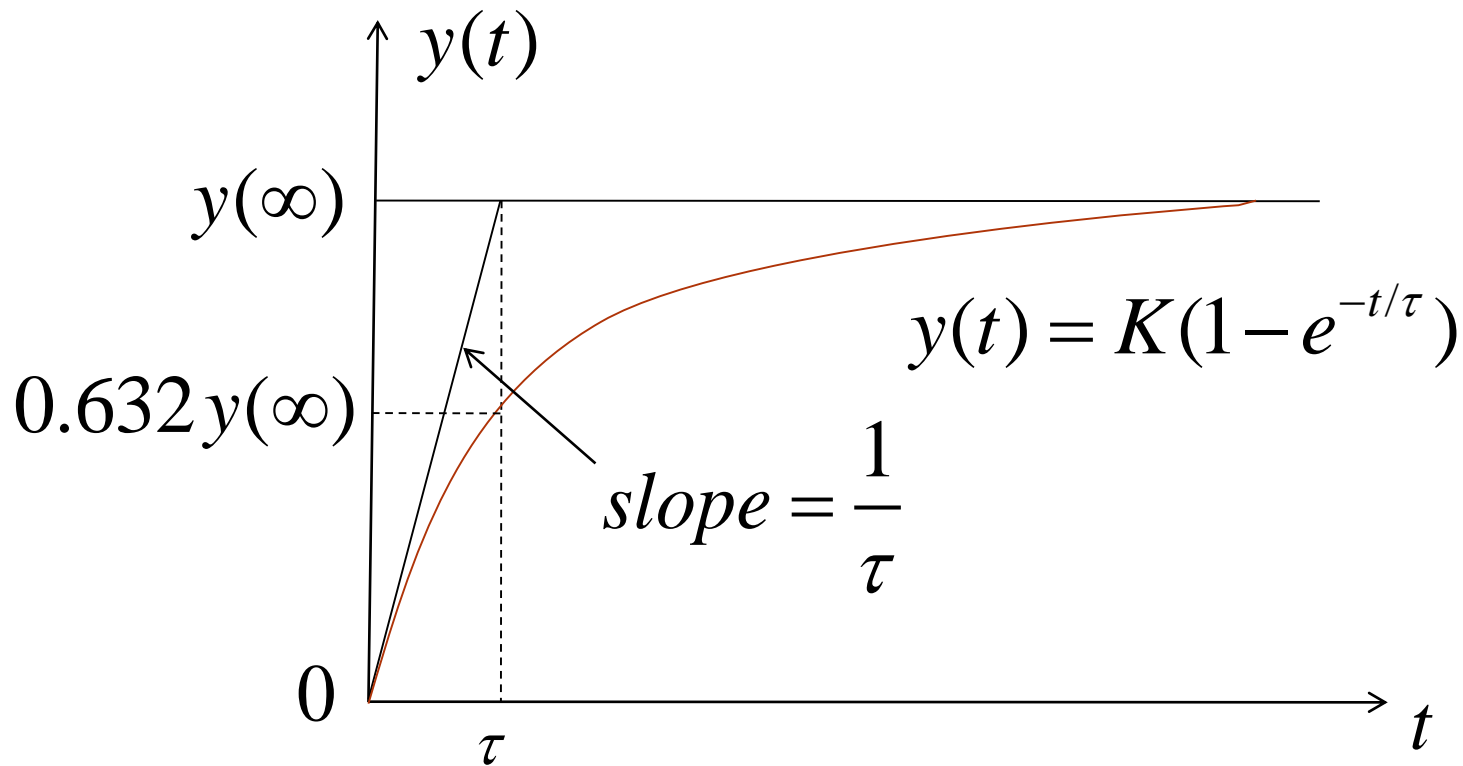
-- judge whether a system is first-order or not

b) The initial slope of the response curve is  $1/T$  and the slope decreases with time

$$\left. \frac{dy(t)}{dt} \right|_{t=0} = \frac{1}{\tau}, \quad \left. \frac{dy(t)}{dt} \right|_{t=\tau} = 0.368 \frac{1}{\tau}, \quad \left. \frac{dy(t)}{dt} \right|_{t=\infty} = 0.$$

It is also a common method to obtain the time constant through the initial slope in control engineering.

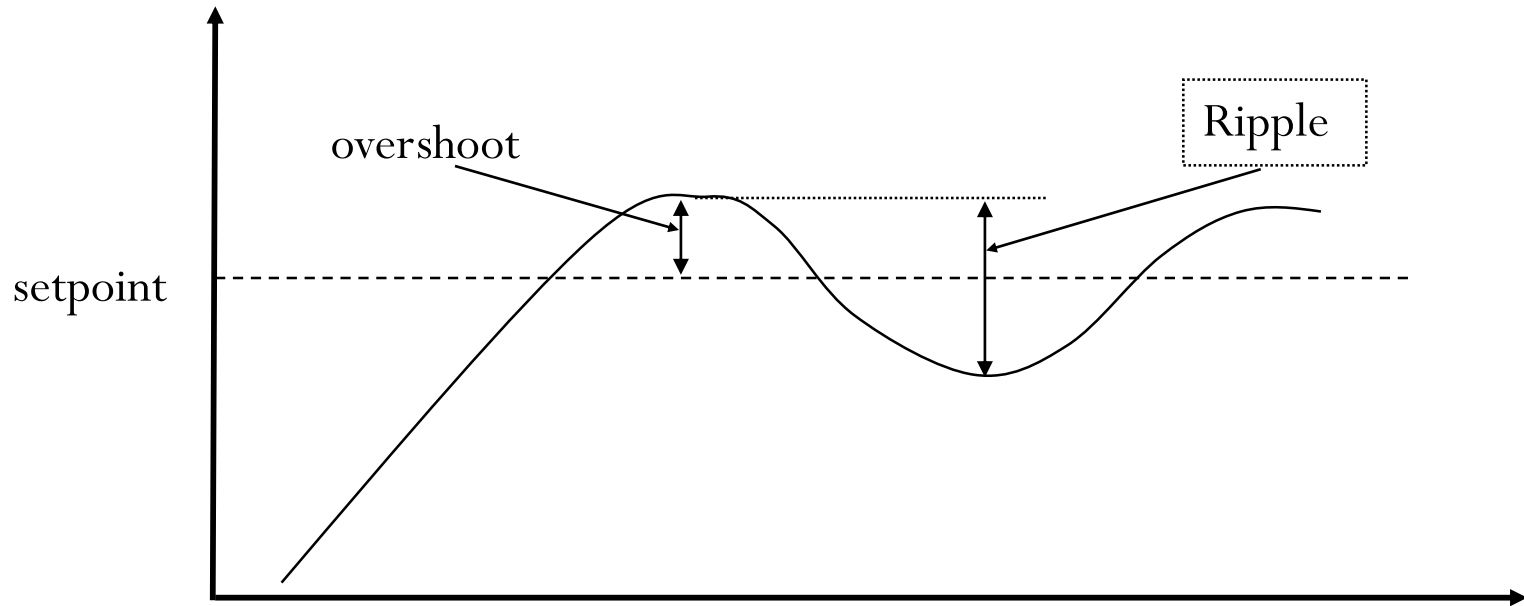
## Unit-step response of a first-order system



# Introduction

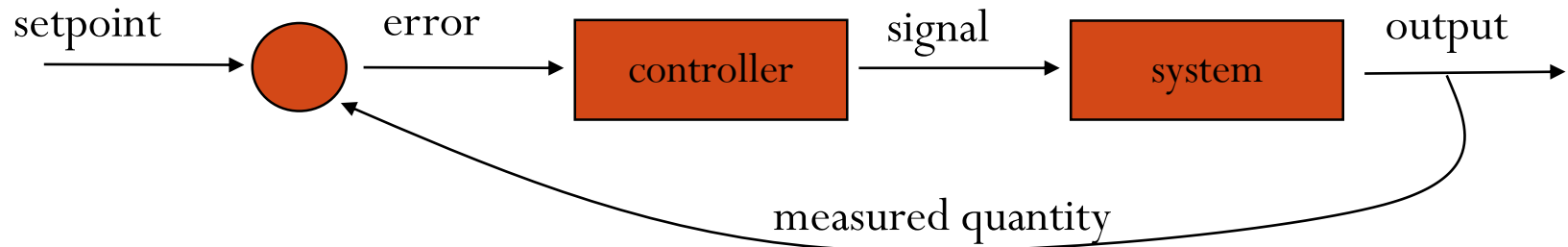
- Suppose you have a system that needs to be controlled
- Your software gives commands, the system responds to it
  - Turn  $x$  degrees to the right
  - Move forward 15 wheel rotations
- Can you always trust your commands will be executed accurately?

# Problem example



- Increase the quantity until you get to the setpoint
  - Temperature, angle, speed, etc
- If too much, reduce the quantity, until the setpoint

# Closed loop controller



- closed loop because it has feedback
- output is measured at a certain frequency
- signal is generated at a certain frequency
- which frequency is not smaller?

# On-off control

- For some systems, on-off signaling is sufficient
- For example, a thermostat, when the heater is either on or off, and early cruise-control systems
  - Could do airflow or speed control also
  - More modern systems do it
- Depending on the frequency of control, overhead of on-off, etc, this could cause overshoots and undershoots (ripples)
  - Oscillation is a common behavior in control systems
  - Need to avoid it at all costs... well, almost all costs

# Proportional control

- Good alternative to on-off control: more “control” 😊
- Signal becomes proportional to the error
  - $P ( \textit{setpoint} - \textit{output} )$
  - Example, car speed for cruise control
- Need to find out value of constant  $P$ 
  - Tuning the controller is a **hard** job
  - If  $P$  is too high, what happens?
  - If  $P$  is too low, what happens?
- Typically a prop cntrl decreases response time (quickly gets to the setpoint) but increases overshoot

# Adding derivative control

- To avoid (or reduce) overshoot/ripple, take into account how fast are you approaching the setpoint
  - If very fast, overshoot may be forthcoming: reduce the signal recommended by the proportional controller
  - If very slow, may never get to setpoint: increase the signal
  - In general:  $D$  (*current measure* – *previous measure*)
- PD controllers are slower than P, but less oscillation, **smaller** overshoot/ripple

# Summary

- Different types of controllers
- PID hardest task is tuning

Controller	Response time	Overshoot	Error
On-off	Smallest	Highest	Large
Proportional	Small	Large	Small
Integral	Decreases	Increases	Zero
Derivative	Increases	Decreases	Small change



Thank you!

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