AKSHAYA INSTITUTE OF TECHNOLOGY

Approved by AICTE, New Delhi, Affiliated to VTU, Belgaum, Recognized by Govt. of Karnataka,
Obalapura Post, Lingapura, Koratagere Road, Tumkur - 572 106, Karnataka

Lecture Notes on

SUBECT : DISCRETE MATHEMATICAL STRUCTURES

SUBECT CODE : BCS405A

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AKSHAYA INSTITUTE OF TECHNOLOGY

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DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING

VISION

To empower the students to be technically competent, innovative and self-motivated with human values and contribute significantly towards betterment of society and to respond swiftly to the challenges of the changing world.

MISSION

M1: To achieve academic excellence by imparting in-depth and competitive knowledge to the students through effective teaching pedagogies and hands on experience on cutting edge technologies.

M2: To collaborate with industry and academia for achieving quality technical education and knowledge transfer through active participation of all the stake holders.

M3: To prepare students to be life-long learners and to upgrade their skills through Centre of Excellence in the thrust areas of Computer Science and Engineering.

Program Specific Outcomes (PSOs)

After Successful Completion of Computer Science and Engineering Program Students will be able to

* Apply fundamental knowledge for professional software development as well as to acquire new skills.
* Implement disciplinary knowledge in problem solving, analyzing and decision-making abilities through different domains like database management, networking, algorithms, and programming as well as research and development.
* Make use of modern computer tools for creating innovative career paths, to become an entrepreneur or desire for higher studies.

Program Educational Objectives (PEOs)

PEO1: Graduates expose strong skills and abilities to work in industries and research organizations.

PEO2 : Graduates engage in team work to function as responsible professional with good ethical behavior and leadership skills.

PEO3: Graduates engage in life-long learning and innovations in multi disciplinary areas.

DISCRETE MATHEMATICAL STRUCTURES		Semester	IV
Course Code	**BCS405A**	CIE Marks	50
Teaching Hours/Week (L:T:P:S)	2:2:0:0	SEE Marks	50
Total Hours of Pedagogy	40	Total Marks	100
Credits	03	Exam Hours	03
Examination type (SEE)	Theory		

Course objectives:

1. To help students to understand discrete and continuous mathematical structures.
2. To impart basics of relations and functions.
3. To facilitate students in applying principles of Recurrence Relations to find the generating functions and solve the Recurrence relations.
4. To have the knowledge of groups and their properties to understand the importance of algebraic properties relative to various number systems.

Teaching-Learning Process

Pedagogy (General Instructions):

These are sample Strategies, teachers can use to accelerate the attainment of the various course outcomes.

1. In addition to the traditional lecture method, different types of innovative teaching methods may be adopted so that the delivered lessons shall develop students' theoretical and applied Mathematical skills.
2. State the need for Mathematics with Engineering Studies and Provide real-life examples.
3. Support and guide the students for self–study.
4. You will assign homework, grading assignments and quizzes, and documenting students' progress.
5. Encourage the students to group learning to improve their creative and analytical skills.
6. Show short related video lectures in the following ways:
   - As an introduction to new topics (pre-lecture activity).
   - As a revision of topics (post-lecture activity).
   - As additional examples (post-lecture activity).
   - As an additional material of challenging topics (pre-and post-lecture activity).
   - As a model solution for some exercises (post-lecture activity).

Module-1: Fundamentals of Logic

Basic Connectives and Truth Tables, Logic Equivalence – The Laws of Logic, Logical Implication – Rules of Inference. The Use of Quantifiers, Quantifiers, Definitions and the Proofs of Theorems.

(8 hours)

(RBT Levels: L1, L2 and L3)

Module-2: Properties of the Integers

Mathematical Induction, The Well Ordering Principle – Mathematical Induction, Recursive Definitions.

Fundamental Principles of Counting: The Rules of Sum and Product, Permutations, Combinations – The Binomial Theorem, Combinations with Repetition. **(8 Hours)**

(RBT Levels: L1, L2 and L3)

Module-3: Relations and Functions

Cartesian Products and Relations, Functions – Plain and One-to-One, Onto Functions. The Pigeon-hole Principle, Function Composition and Inverse Functions.

Properties of Relations, Computer Recognition – Zero-One Matrices and Directed Graphs, Partial Orders – Hasse Diagrams, Equivalence Relations and Partitions. **(8 hours)**

(RBT Levels: L1, L2 and L3)

Module-4: The Principle of Inclusion and Exclusion

The Principle of Inclusion and Exclusion, Generalizations of the Principle, Derangements – Nothing is in its Right Place, Rook Polynomials.

Recurrence Relations: First Order Linear Recurrence Relation, The Second Order Linear Homogeneous Recurrence Relation with Constant Coefficients. **(8 Hours)**

(RBT Levels: L1, L2 and L3)

Module-5: Introduction to Groups Theory

Definitions and Examples of Particular Groups Klein 4-group, Additive group of Integers modulo n, Multiplicative group of Integers modulo-p and permutation groups, Properties of groups, Subgroups, cyclic groups, Cosets, Lagrange's Theorem. **(8 Hours)**

(RBT Levels: L1, L2 and L3)

Course outcome (Course Skill Set)

At the end of the course, the student will be able to:

1. Apply concepts of logical reasoning and mathematical proof techniques in proving theorems and statements.
2. Demonstrate the application of discrete structures in different fields of computer science.
3. Apply the basic concepts of relations, functions and partially ordered sets for computer representations.
4. Solve problems involving recurrence relations and generating functions.
5. Illustrate the fundamental principles of Algebraic structures with the problems related to computer science & engineering.

Assessment Details (both CIE and SEE)

The weightage of Continuous Internal Evaluation (CIE) is 50% and for Semester End Exam (SEE) is 50%. The minimum passing mark for the CIE is 40% of the maximum marks (20 marks out of 50) and for the SEE, the minimum passing mark is 35% of the maximum marks (18 out of 50 marks). The student is declared as a pass in the course if he/she secures a minimum of 40% (40 marks out of 100) in the sum total of the CIE (Continuous Internal Evaluation) and SEE (Semester End Examination) taken together.

Continuous Internal Evaluation:

- There are 25 marks for the CIE's Assignment component and 25 for the Internal Assessment Test component.
- Each test shall be conducted for 25 marks. The first test will be administered after 40-50% of the coverage of the syllabus, and the second test will be administered after 85-90% of the coverage of the syllabus. The average of the two tests shall be scaled down to 25 marks
- Any two assignment methods mentioned in the 22OB2.4, if an assignment is project-based then only one assignment for the course shall be planned. The schedule for assignments shall be planned properly by the course teacher. The teacher should not conduct two assignments at the end of the semester if two assignments are planned. Each assignment shall be conducted for 25 marks. (If two assignments are conducted then the sum of the two assignments shall be scaled down to 25 marks)

The final CIE marks of the course out of 50 will be the sum of the scale-down marks of tests and assignment/s marks.

The Internal Assessment Test question paper is designed to attain the different levels of Bloom's taxonomy as per the outcome defined for the course.

Semester-End Examination:

Theory SEE will be conducted by the University as per the scheduled timetable, with common question papers for the course (**duration 03 hours).**

1. The question paper will have ten questions. Each question is set for 20 marks.
2. There will be 2 questions from each module. Each of the two questions under a module (with a maximum of 3 sub-questions), **should have a mix of topics** under that module.
3. The students have to answer 5 full questions, selecting one full question from each module.

Marks scored shall be proportionally reduced to 50 marks

Suggested Learning Resources:

Books (Name of the author/Title of the Book/Name of the publisher/Edition and Year)

Text Books:

1. **Ralph P. Grimaldi, B V Ramana: "Discrete Mathematical Structures an Applied Introduction",** 5th Edition, Pearson Education, 2004.
2. **Ralph P. Grimaldi: "Discrete and Combinatorial Mathematics"**, 5th Edition, Pearson Education. 2004.

Reference Books:

1. **Basavaraj S Anami and Venakanna S Madalli: "Discrete Mathematics – A Concept-based approach"**, Universities Press, 2016
2. **Kenneth H. Rosen: "Discrete Mathematics and its Applications"**, 6th Edition, McGraw Hill, 2007.
3. **Jayant Ganguly: "A Treatise on Discrete Mathematical Structures",** Sanguine-Pearson, 2010.
4. **D.S. Malik and M.K. Sen: "Discrete Mathematical Structures Theory and Applications,** Latest Edition, Thomson, 2004.
5. **Thomas Koshy: "Discrete Mathematics with Applications"**, Elsevier, 2005, Reprint 2008.

Web links and Video Lectures (e-Resources):

- http://nptel.ac.in/courses.php?disciplineID=111
- http://www.class-central.com/subject/math(MOOCs)
- http://academicearth.org/
- VTU e-Shikshana Program
- VTU EDUSAT Program.
- http://www.themathpage.com/
- http://www.abstractmath.org/
- http://www.ocw.mit.edu/courses/mathematics/

Activity-Based Learning (Suggested Activities in Class)/Practical-Based Learning

- Quizzes
- Assignments
- Seminar

Sub:- Discrete Mathematical structure.

Subcode: BCS405A

Module-1: Fundementals of Logic.

→ Basic connectives and Truth tables

→ Logic Equivalence - The Laws of Logic

→ Logical Implication - Rules of Inference

→ The Use of Quantifiers, Quantifiers

→ Definitions and proofs of Theorems

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FUNDEMENTALS OF LOGIC

Proposition :- It is a statement or declaration which in a given content, can be either true or false but not both.

Ex: 1. Banglore is in karnataka (True)
2. Three is a prime number (True)
3. Seven is divisible by 3. (False)
4. Every rectangle is a square. (False)

Note :- All sentences are not propositions

Ex: 1. Consider a triangle ABC
2. $xy = yx$
3. what an amazing day.

Truth value :- The truth or falsity of a proposition is called its truth value.

If proposition, p is true, its truth value is 1, if p is false, its truth value is '0'

Logical connectives :- The new propositions are obtained by using phrases like not, or, and, if then, if and only if etc. Such words or phrases are called logical connectives.

Compound propositions :- The new propositions obtained by the use of logical connectives are called compound propositions.

The original propositions from which a compound propositions is obtained are called the components or the primitive of the compound propositions.

Simple propositions :- propositions that donot contain any logical connective are called simple propositions.

Negation :- A proposition obtained by inserting the word 'not' at an appropriate place in a given proposition is called the Negation of the given proposition. It is denoted as $\neg p$

Ex: p : 3 is an odd number
$\neg p$: 3 is not an odd number

Truth table for Negation.

p	$\neg p$
0	1
1	0

p: I like Mathematics
$\sim p$: I don't like mathematics

Conjuction :- A compound proposition obtained by combining two given propositions with and in between them is called the conjuction of the given propositions. It is denoted by $p \wedge q$

Ex: p: $\sqrt{2}$ is an irrational number
I like Mathematics
q: $2+5=7$
I score good marks in mathematics

$p \wedge q$: $\sqrt{2}$ is an irrational number and $2+5=7$

Truth table for conjuction.

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Disjunction :- A compound proposition obtained by combining two given propositions by inserting 'or' in between them is called the disjunction of the given propositions. It is denoted as $p \vee q$.

Ex: p : $\sqrt{2}$ is an irrational number

q : 9 is a prime number

$p \vee q$: $\sqrt{2}$ is an irrational number or 9 is a prime number.

Truth table for disjunction.

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive Disjunction:- If a compound proposition p or q is true, when either p or q is true but not both then it is said to be Inclusive disjunction denoted by $p \veebar q$

Ex: p : $\sqrt{2}$ is an irrational number

q : $2+3=5$

$p \veebar q$: Either $\sqrt{2}$ is an irrational number or $2+3=5$, but not both

Truth table for exclusive disjunction

p	q	$p \veebar q$
0	0	0
0	1	1
1	0	1
1	1	0

Conditional (Implication)

A compound proposition obtained by combining two propositions by the words 'if' and 'then' at appropriate places is called a Conditional or an Implication. It is denoted as $p \rightarrow q$ (if p then q)

Ex: p : I weigh more than 120 pounds

q : I shall enroll in an excercise class

$p \rightarrow q$: If I weigh more than 120 pounds then I shall enroll in an excercese class.

Truth table for conditional

p	q	$p\to q$
0	0	1
0	1	1
1	0	0
1	1	1

Biconditional (Double implication) :-

Let p and q are the two propositions, then the biconditional $p\to q$ the conjuction (and) of the biconditional of p and q and $q\to p$ is called the Biconditional of p and q. It is denoted as $p\leftrightarrow q$

$\therefore\ p\leftrightarrow q \Leftrightarrow (p\to q)\wedge(q\to p)$

$\Rightarrow p\to q$ is read as "If p then q and if q then p"

or "p if and only if q"

Ex: p: I shall enroll in an excercise class.

q: I weigh more than 120 pounds.

$p\leftrightarrow q$: I shall enroll in an excercise class if and only if I weigh more than 120 pounds.

Truth table for Biconditional.

p	q	$p\to q$	$q\to p$	$p\leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

problems:

1. Let p: A circle is a conic
q: $\sqrt{5}$ is a real number.
r: Exponential series is convergent.

Express the following compound proposition in words.

(i) $p \wedge (\sim q)$ (ii) $(\sim p) \vee q$ (iii) $p \underline{\vee} \sim q$

(iv) $q \rightarrow \sim p$ (v) $p \rightarrow (q \underline{\vee} r)$ (vi) $\sim p \leftrightarrow q$

Soln.

(i) $p \wedge (\sim q)$: A circle is a conic and $\sqrt{5}$ is not a real number.

(ii) $(\sim p) \vee q$: A circle is not a conic or $\sqrt{5}$ is a real number.

(iii) $p \underline{\vee} \sim q$: Either a circle is a conic or $\sqrt{5}$ is not a real number.

(iv) $q \rightarrow \sim p$: If $\sqrt{5}$ is a real number then a circle is not a conic.

(v) $p \rightarrow (q \underline{\vee} r)$: If a circle is a conic then either $\sqrt{5}$ is a real number or the exponential series is cgt.

(vi) $\sim p \leftrightarrow q$: If a circle is not a conic then $\sqrt{5}$ is a real number and if $\sqrt{5}$ is a real number then a circle is not a conic.

2 Construct the truth table for the following compound propositions:

(i) $p \wedge \sim q$ (ii) $(\sim p) \vee q$

(iii) $p \to (\sim q)$ (iv) $(\sim p) \underline{\vee} (\sim q)$

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$p \to \sim q$	$(\sim p) \underline{\vee} (\sim q)$
0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	1	0	0	0	1	0	0

3. Let p and q be primitive statements for which the conditional $p \to q$ is false. Determine the truth values of the following compound propositions.

(i) $p \wedge q$ (ii) $\sim p \vee q$ (iii) $q \to p$ (iv) $\sim q \to \sim p$.

Sol^n.

$p \to q$	p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \vee q$	$q \to p$	$\sim q \to \sim p$
0	1	0	0	1	0	0	1	0.

4 construct the truth tables for the following compound propositions:

(i) $(p \vee q) \wedge r$

(ii) $p \vee (q \wedge r)$

Sol:

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$q \wedge r$	$p \vee (q \wedge r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	0	1	1	1	0	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

5. Construct the truth tables for the following compound propositions:

(i) $(p \wedge q) \rightarrow \sim r$ (ii) $q \wedge (\sim r \rightarrow p)$.

Sol:

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \rightarrow (\sim r)$	$(\sim r) \rightarrow p$	$q \wedge (\sim r \rightarrow p)$
0	0	0	1	0	1	0	0
0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	0	1	0	1	1

6. Let p, q and r be propositions having truth values 0, 0 and 1 respectively. Find the truth values of the following compound propositions.

(i) $(p \vee q) \vee r$

p	q	r	$p \vee q$	$(p \vee q) \vee r$
0	0	1	0	1

(ii) $(p \wedge q) \wedge r$

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$
0	0	1	0	0

(iii) $(p \wedge q) \rightarrow r$

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
0	0	1	0	1

(iv) $p \rightarrow (q \wedge r)$

p	q	r	$q \wedge r$	$p \rightarrow (q \wedge r)$
0	0	1	0	1

(v) $p \wedge (r \rightarrow q)$

p	q	r	$r \rightarrow q$	$p \wedge (r \rightarrow q)$
0	0	1	0	0

(vi) $p \rightarrow (q \rightarrow \sim r)$

p	q	r	$\sim r$	$q \rightarrow \sim r$	$p \rightarrow (q \rightarrow \sim r)$
0	0	1	0	1	1

7 Give the conjuction and disjunction of p and q in the following cases : in each case indicate the truth value.

(1) p : 4 is a perfect square

q : 27 is a prime number.

(2) p : 5 is divisible by 2

q : 7 is a multiple of 5.

Sol?.

(1) Conjuction of $p \wedge q \Rightarrow$ 4 is a perfect square and 27 is a prime number.

$p = 1$ and $q = 0 \Rightarrow p \wedge q$ is false.

Disjunction of $p \vee q \Rightarrow$ 4 is a perfect square or 27 is a prime number.

$p = 1 \quad q = 0 \rightarrow p \vee q$ is true

(2) Conjunction of $p \wedge q \Rightarrow$ 5 is divisible by 2 and 7 is a multiple of 5.

$p = 0$ & $q = 0 \Rightarrow p \wedge q = 0$ or false.

Disjunction of $p \vee q \Rightarrow$ 5 is divisible by 2 or 7 is a multiple of 5.

$p = 0, \ q = 0 \Rightarrow p \vee q = 0$

8. Given that p is true and q is false, find the truth values of the following.

(i) $(\sim p) \wedge q$

p	q	$\sim p$	$\sim p \wedge q$
1	0	0	0

(ii) $\sim(p \wedge q) \vee \{\sim(q \leftrightarrow p)\}$

p	q	$p \wedge q$	$\overset{x}{\sim(p \wedge q)}$	$q \leftrightarrow p$	$\overset{y}{\sim(q \leftrightarrow p)}$	$x \vee y$
1	0	0	1	0.	1	1

(iii) $\sim(p \rightarrow (\sim q))$.

p	q	$\sim q$	$p \rightarrow \sim q$	$\sim[p \rightarrow \sim q]$
1	0	1	1	0

(iv) $(p \rightarrow q) \vee \{\sim(p \leftrightarrow \sim q)\}$.

p	q	$\overset{x}{p \rightarrow q}$	$\sim q$	$p \leftrightarrow \sim q$	$\overset{y}{\sim(p \leftrightarrow \sim q)}$	$x \vee y$
1	0	0	1	1	0	0

(v) $(p \wedge q) \rightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
1	0	0	1	1

(vi) $[p \rightarrow (\sim q)] \vee \{q \rightarrow (\sim p)\}$.

p	q	$\sim q$	$\overset{x}{p \rightarrow \sim q}$	$\sim p$	$\overset{y}{q \rightarrow \sim p}$	$x \vee y$
1	0	1	1	0	1	1

9. Construct the truth table of $p \to (q \to r)$

p	q	r	$q \to r$	$p \to (q \to r)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
1	0	0	1	1
0	1	1	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

10. Construct the truth table of $(p \to q) \to r$

p	q	r	$p \to q$	$(p \to q) \to r$
0	0	0	1	0
0	0	1	1	1
0	1	0	1	0
1	0	0	0	1
0	1	1	1	1
1	0	1	0	$\cancel{0}$ 1
1	1	0	1	0
1	1	1	1	1

Assignments :-

1. Construct the truth tables for the following

(i) $p \vee (\sim q)$

(ii) $p \wedge (q \wedge p)$

(iii) $p \wedge (q \vee p)$

(iv) $(p \vee q) \wedge (\sim p)$

(v) $\sim (p \vee \sim q)$

(vi) $q \leftrightarrow (\sim p \vee \sim q)$

(vii) $[(p \wedge q) \vee \sim r] \leftrightarrow p$.

2. Consider the following propositions concerned with a certain triangle ABC

p : ABC is isosceles

q : ABC is equilateral

r : ABC is equiangular

write down the following propositions in words

(1) $p \wedge \sim q$

(2) $\sim p \vee q$

(3) $p \rightarrow q$

(4) $q \rightarrow p$

(5) $\sim r \rightarrow \sim q$

(6) $p \leftrightarrow \sim q$.

Tautology and Contradiction

Tautology :- A compound proposition which is always true regardless of the truth value of its components is called a tautology.

Contradiction :- A compound proposition which is always false regardless of the truth value of its components is called a contradiction or absurdity.

Contigency :- A compound proposition that can be true or false is called a contigency.

OR

Contigency is a compound proposition which is neither tautology not a contradiction.

Problems :-

1. Prove that the compound proposition $p \vee \sim p$ is a tautology and $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \wedge \sim p$	$p \vee \sim p$
0	1	0	1
1	0	0	1

$\Rightarrow$ $p \vee \sim p$ is always true, hence it is tautology & $p \wedge \sim p$ is always false, hence it is a contradiction.

2. Show that, for any propositions p and q, the compound proposition $p \to (p \vee q)$ is a tautology and the compound proposition $p \wedge (\sim p \wedge q)$ is a contradiction.

p	q	$p \vee q$	$p \to (p \vee q)$	$\sim p$	$\sim p \wedge q$	$p \wedge (\sim p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0.
			Tautology			Contradiction

3. Show that the truth values of the following compound proposition are independent of the truth values of their components.

(i) $\{p \wedge (p \to q)\} \to q$

(ii) $(p \to q) \leftrightarrow (\sim p \vee q)$

(i)

p	q	$p \to q$	$p \wedge (p \to q)$	$r \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

(ii)

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$u \leftrightarrow v$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

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Logical equivalence

Two propositions u and v are said to be Logically equivalent whenever u and v have the same truth value or the biconditional $u \leftrightarrow v$ is always a tautology.

The logical equivalence is denoted by $\Leftrightarrow$

Problems:-

1. For any two propositions p, q prove that $(p \to q) \Leftrightarrow (\sim p) \vee q$

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
0	0	1	1	1
0	1	1	1	0
1	0	0	0	1
1	1	1	0	1.

$\therefore$ $p \to q$ and $\sim p \vee q$ have the same truth values for all possible values of p and q.

$\therefore$ $(p \to q) \Leftrightarrow (\sim p) \vee q$

2. prove that, for any propositions p and q, the compound propositions $p \underline{\vee} q$ and $(p \vee q) \wedge (\sim p \vee \sim q)$ are Logically equivalent.

p	q	$p \vee q$ (x)	$p \veebar q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$ (y)	$x \wedge y$
0	0	0	0	1	1	1	0
0	1	1	1	1	0	1	1
1	0	1	1	0	1	1	1
1	1	1	0	0	0	0	0

$$\therefore (p \veebar q) \Longleftrightarrow (p \vee q) \wedge (\sim p \vee \sim q)$$

3. prove that, for any three propositions p, q, r

$$[p \to (q \wedge r)] \Longleftrightarrow [(p \to q) \wedge (p \to r)]$$

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$ (x)	$p \to r$ (y)	$x \wedge y$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
1	0	0	0	1	0	0	0
0	1	1	1	0	1	0	0
1	0	1	0	0	0	1	0
1	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1

$$\therefore [p \to (q \wedge r)] \Longleftrightarrow [(p \to q) \wedge (p \to r)]$$

4. V.Imp Prove that, for any three propositions p, q, r

$$[(p \vee q) \rightarrow r] \Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$$ (Dec 2010)

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	$\overset{x}{p \rightarrow r}$	$\overset{y}{q \rightarrow r}$	$x \wedge y$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
1	0	0	1	0	0	1	0
0	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

5 Examine whether $[(p \vee q) \rightarrow r] \leftrightarrow [\sim r \rightarrow \sim (p \vee q)]$ is a tautology.

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	$\sim r$	$\sim (p \vee q)$	$\sim r \rightarrow \sim (p \vee q)$
0	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1
0	1	0	1	0	1	0	0
1	0	0	1	0	1	0	0
0	1	1	1	1	0	0	1
1	0	1	1	1	0	0	1
1	1	0	1	0	1	0	0
1	1	1	1	1	0	0	1

$\therefore [(p \vee q) \rightarrow r] \Leftrightarrow [\sim r \rightarrow \sim (p \vee q)]$

Laws of Logic :-

We recall that if $P \equiv Q$, then the propositions P and Q are said to be logically equivalent. This equivalence is called law of Logic.

Sl. No.	Law	Name of the law.
1.	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative law
2.	$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associative law
3.	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive law
4.	$\sim(p \vee q) \equiv \sim p \wedge \sim q$ $\sim(p \wedge q) \equiv \sim p \vee \sim q$	De-Morgan's law
5.	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption law
6	$\sim(\sim p) \equiv p$	Double negation law
7	$p \vee p \equiv p$, $p \wedge p \equiv p$	Idempotent law
8	$\sim(p \rightarrow q) \equiv p \wedge \sim q$	Negation of condition
9	$p \wedge T \equiv p$; $p \vee F \equiv p$	Identity law
10.	$p \vee T \equiv T$, $p \wedge F \equiv F$	Domination law
11.	$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$	Inv. law

Duality of proposition :-

Two propositions p & q involving basic connectives $\vee, \wedge, \sim$ are said to be duals of each other if replacement of $\vee$ by $\wedge$, and $\wedge$ by $\vee$ ('$\sim$' remains unchanged). Further if T by F and F by T

Note :-

1. $p \underline{\vee} q \Leftrightarrow (p \vee q) \wedge (\sim p \vee \sim q)$
2. $p \rightarrow q \Leftrightarrow \sim p \vee q$
3. $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

THE LAWS OF LOGIC :-

For any primitive statements p, q, r any tautology T_0, and any contradiction F_0, the following Laws hold good.

1. Law of double negation

$$\sim\sim p \Leftrightarrow p$$

2. Idempotent Laws

$$(p \vee p) \Leftrightarrow p$$

$$(p \wedge p) \Leftrightarrow p.$$

3. Identity Laws

$$(p \vee F_0) \Leftrightarrow p$$

$$(p \wedge T_0) \Leftrightarrow p$$

4. Inverse laws

$$(p \vee \sim p) \Leftrightarrow T_0$$

$$(p \wedge \sim p) \Leftrightarrow F_0$$

5. Domination Laws

$$(p \vee T_0) \Leftrightarrow T_0$$

$$(p \wedge F_0) \Leftrightarrow F_0$$

6. Commutative Laws

$$(p \vee q) \Leftrightarrow (q \vee p)$$

$$(p \wedge q) \Leftrightarrow (q \wedge p)$$

7. Absorption Laws

$$[p \vee (p \wedge q)] \Leftrightarrow p$$

$$[p \wedge (p \vee q)] \Leftrightarrow p$$

8. De Morgan Laws:-

$$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

$$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$$

9. Associative laws:-

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

$$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

10. Distributive laws:-

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

[prove all the laws using truth table]

Some important relations

i) $p \rightarrow q \Leftrightarrow \sim p \vee q$

ii) $\sim(p \rightarrow q) \Leftrightarrow p \wedge \sim q$

Transitive and Substitution Rules

(1) If u, v, w are propositions such that $u \Leftrightarrow v$ and $v \Leftrightarrow w$, then $u \Leftrightarrow w$ (This is known as the transitive rule).

(2) Suppose that a compound proposition u is a tautology and p is a component of u. If we replace each occurence of p in u by a proposition q, then the resulting compound proposition v is also a tautology (This is called a substitution Rule).

(3) Suppose that 'u' is a compound proposition which contains a component p. Let q be a proposition that $q \Leftrightarrow p$. Suppose we replace one or more occurence of p by q and obtain a compound proposition v. Then $v \Leftrightarrow u$. (This is known as Substitution Rule)

Problems:-

1. Let 'x' be a specified number. write down the negation of the following conditional:
'If 'x' is an integer then x is a rational number'

Sol^n: Given: $p \to q$

where, p: x is an integer
q: x is a rational number.

$\Rightarrow \sim(p \to q) \equiv p \wedge \sim q$

$\therefore$ "x is an integer and x is not a rational no"

2. Let 'x' be a specified number. write down the negation of the following proposition:
"If 'x' is not a real number, then it is not a rational number and not an irrational number"

Sol^n: Let p: x is a real number
q: x is a rational number
r: x is an irrational number.

$\Rightarrow \sim p \to \sim q \wedge \sim r$

$\therefore \sim[\sim p \to (\sim q \wedge \sim r)] \equiv \sim p \wedge \sim(\sim q \wedge \sim r)$

$\equiv \sim p \wedge (q \vee r)$

Thus the negation of the given proposition is "x is not a real number and it is a rational number or an irrational number."

3. Simplify the following compound propositions using the Laws of Logic:

(i) $(p \vee q) \wedge \sim\{(\sim p) \vee q\}$

(ii) $\sim[\sim\{(p \vee q) \wedge r\} \vee \sim q]$

Sol^n. (i) $(p \vee q) \wedge \sim\{(\sim p) \vee q\}$

$\equiv (p \vee q) \wedge (p \wedge \sim q)$

$\equiv \{(p \vee q) \wedge p\} \wedge \sim q$ using Associative law

$\equiv \{p \wedge (p \vee q)\} \wedge \sim q$ using commutative law

$\equiv p \wedge \sim q$ using Absorption Law.

(ii) $\sim[\sim\{(p \vee q) \wedge r\} \vee \sim q]$

$\equiv \sim[\sim\{(p \vee q) \wedge r\}] \wedge q$ (Demorgans law)

$\equiv ((p \vee q) \wedge r) \wedge q$ (Double negation)

$\equiv (p \vee q) \wedge (q \wedge r)$ (Associative law & comm. law)

$\equiv \{(p \vee q) \wedge q\} \wedge r$ (Asso. law)

$\equiv q \wedge r$ (Absorption law)

4. prove the following Logical equivalences without using truth tables:

(i) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$

(ii) $[p \vee q \vee (\sim p \wedge \sim q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

(iii) $[(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

Sol^n (i) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p \vee p$ (by Absorption)

$\Leftrightarrow p$ (by Idempotent)

(ii) $\sim p \wedge \sim q \wedge r \Leftrightarrow (\sim p \wedge \sim q) \wedge r$ (by Associative)

$\Leftrightarrow \sim(p \vee q) \wedge r$ (by Demorgans)

$\therefore$ $p \vee q \vee (\sim p \wedge \sim q \wedge r) \Leftrightarrow (p \vee q) \vee [\sim(p \vee q) \wedge r]$

$\Leftrightarrow [(p \vee q) \vee \sim(p \vee q)] \wedge [(p \vee q) \vee r]$ (by Dist)

$\Leftrightarrow T_0 \wedge (p \vee q \vee r)$ (Inv & Ass)

$\Leftrightarrow p \vee q \vee r$ (comm & Identity)

(iii) $(\sim p \vee \sim q) \to (p \wedge q \wedge r)$ $[\because u \to v \Leftrightarrow \sim u \vee v]$

$\Leftrightarrow \sim(\sim p \vee \sim q) \vee (p \wedge q \wedge r)$

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$ (Demorgans & Associative)

$\Leftrightarrow p \wedge q$ (by Absorption)

5. prove the following Logical equivalences:

(i) $[(p \vee q) \wedge (p \vee \sim q)] \vee q \Leftrightarrow p \vee q$

(ii) $(p \to q) \wedge [\sim q \wedge (r \vee \sim q)] \Leftrightarrow \sim(q \vee p)$

Soln: (i) $(p \vee q) \wedge (p \vee \sim q)$

$\Leftrightarrow p \vee (q \wedge \sim q)$ (by Distributive)

$\Leftrightarrow p \vee F_0$, $(\because q \wedge \sim q$ is a contradiction)

$\Leftrightarrow p$ (by Identity)

$\therefore [(p \vee q) \wedge (p \vee \sim q)] \vee q \Leftrightarrow p \vee q$

(ii) $(p \to q) \wedge [\sim q \wedge (r \vee \sim q)]$

$\Leftrightarrow (p \to q) \wedge [\sim q \wedge (\sim q \vee r)]$ (by comm)

$\Leftrightarrow (p \to q) \wedge \sim q$ (by Absorption)

$\Leftrightarrow \sim[(p \to q) \to q]$ $(\because \sim(u \to v) \Leftrightarrow u \wedge \sim v)$

$\Leftrightarrow \sim[\sim(p \to q) \vee q]$ $(\because u \to v \Leftrightarrow \sim u \vee v)$

$\Leftrightarrow \sim[(p \wedge \sim q) \vee q]$

$\Leftrightarrow \sim[q \vee (p \wedge \sim q)]$ (by comm)

$\Leftrightarrow \sim[(q \vee p) \wedge (q \vee \sim q)]$ (by Distributive)

$\Leftrightarrow \sim[(q \vee p) \wedge T_0]$ $(\because q \vee \sim q$ is a tautology)

$\Leftrightarrow \sim(q \vee p)$ (by Identity)

6. prove the following

(i) $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

(ii) $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

Sol^n: (i) $p \to (q \to r) \Leftrightarrow \sim p \vee (\sim q \vee r)$

$\Leftrightarrow (\sim p \vee \sim q) \vee r$

$\Leftrightarrow \sim (p \wedge q) \vee r$

$\Leftrightarrow (p \wedge q) \to r.$

(ii) $[\sim p \wedge (\sim q \wedge r)] \Leftrightarrow (\sim p \wedge \sim q) \wedge r$

$\Leftrightarrow [\sim (p \vee q)] \wedge r$

$\Leftrightarrow r \wedge [\sim (p \vee q)]$

and $(q \wedge r) \vee (p \wedge r) \Leftrightarrow (r \wedge q) \vee (r \wedge p)$

$\Leftrightarrow r \wedge (q \vee p)$

$\Leftrightarrow r \wedge (p \vee q)$

$\therefore [\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)]$

$\Leftrightarrow \{r \wedge [\sim (p \vee q)]\} \vee \{r \wedge (p \vee q)\}$

$\Leftrightarrow r \wedge \{[\sim (p \vee q)] \vee (p \vee q)\}$

$\Leftrightarrow r \wedge T_0$ $\quad$ ($\because$ $[\sim (p \vee q)] \vee (p \vee q)$ is always true)

$\Leftrightarrow r$

7. prove the following result:

$\sim [\{(p \vee q) \wedge r\} \to \sim q] \Leftrightarrow \sim [\sim [(p \vee q) \wedge r] \vee \sim q]$

$\Leftrightarrow q \wedge r$

Sol^n: $\sim [\{(p \vee q) \wedge r\} \to \sim q] \Leftrightarrow \sim [\sim \{(p \vee q) \wedge r\} \vee \sim q]$ —— ①

$\Leftrightarrow \sim\sim [\{(p \vee q) \wedge r\} \wedge q]$

$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$

$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$

$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$

$\Leftrightarrow [q \wedge (q \vee p)] \wedge r$

$\Leftrightarrow q \wedge r$ —— (II)

(I) & (II) LHS = RHS.

8. prove that

$[(P \vee Q) \wedge \sim\{\sim P \wedge (\sim Q \vee \sim R)\}] \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$

is a tautology.

Sol^n: $w \equiv u \vee v$

$u \equiv (P \vee Q) \wedge \sim\{\sim P \wedge (\sim Q \vee \sim R)\}$

& $v \equiv (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$.

$u \Leftrightarrow (P \vee Q) \wedge \sim\{\sim P \wedge \sim(Q \wedge R)\}$

$\Leftrightarrow (P \vee Q) \wedge \{P \vee (Q \wedge R)\}$

$\Leftrightarrow P \vee \{Q \wedge (Q \wedge R)\}$

$\Leftrightarrow P \vee \{(Q \wedge Q) \wedge R\}$

$\Leftrightarrow P \vee (Q \wedge R)$

& $v \Leftrightarrow \sim(P \vee Q) \vee \sim(P \vee R)$

$\Leftrightarrow \sim\{(P \vee Q) \wedge (P \vee R)\}$

$\Leftrightarrow \sim\{P \vee (Q \wedge R)\}$

$\Leftrightarrow \sim u$

$\therefore w \equiv u \vee v \Leftrightarrow u \vee \sim u \Leftrightarrow T_0$.

$\therefore w = u \vee v$ is a tautology.

Converse :-
Inverse :-
Contrapositive :-

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Logical Implication :-

Consider a conditional $p \to q$ where p & q are related in a way that the truth value of q depends on the truth value of p and vice-versa. Such conditionals are called as hypothetical or implicative statements.

When a hypothetical statement $p \to q$ is such that q is true whenever p is true, then we say that 'p logically implies q'. This is symbolically represented as $p \Rightarrow q$ where $\Rightarrow$ denotes implication.

When $p \Rightarrow q$ is true always, then $p \to q$ is a tautology. So in this case we say that the conditional $p \to q$ is a logical implication.

If $p \to q$ is not a tautology, then $p \to q$ is not a logical implication. So we write $p \not\Rightarrow q$ [p does not imply q]. Which means q need not be true when p is true.

Necessary and Sufficient Conditions

Consider two propositions p & q whose truth values are inter related. Then for $p \to q$ to be a logical implication. The following statements hold good.

i) $p \Rightarrow q$

ii) p is sufficient for q

iii) q is necessary for p.

Problems :-

1. Give the necessary and sufficient condition for the following conditionals.

(a) If a quadrilateral is a parallelogram, then its diagonals bisect each other

(b) If a real number x^2 is greater than zero, then x is not equal to zero.

(c) If a triangle is not isosceles then it is not equilateral.

Sol^n: a) p: quadrilateral is a parallelogram

q: Quadrilaterals diagonals bisect each other

∴ The given statement can be symbolically represe-nted as $p \to q$

W.K.T. p is sufficient for q and

q is necessary for p

∴ A necessary condition for a quadrilateral to be a parallelogram is that its diagonals bisect each other.

A sufficient condition for the diagonals of a quadrilateral to bisect each other is that the quadrilateral is a parallelogram.

[Ily for b & c]

2. Write down the contrapositive of $[p \rightarrow (q \rightarrow r)]$ with
(a) only one occurence of the connective $\rightarrow$
(b) No occurence of the connective $\rightarrow$

Soln: contrapositive of $[p \rightarrow (q \rightarrow r)]$ is $[\sim(q \rightarrow r) \rightarrow \sim p]$

$[u \rightarrow v \Leftrightarrow \sim v \rightarrow \sim u]$

$$[\sim(q \rightarrow r) \rightarrow \sim p] \Leftrightarrow \sim \{\sim(q \rightarrow r)\} \vee \sim p$$
$$\Leftrightarrow (q \rightarrow r) \vee \sim p \quad \text{— (1)}$$
$$\Leftrightarrow (\sim q \vee r) \vee \sim p. \quad \text{— (2)}$$

(i) & (ii) are the required representations.

3. prove the following by logical implication
(i) $[p \wedge (p \rightarrow q)] \Rightarrow q$ (ii) $[(p \rightarrow q) \wedge \sim q] \Rightarrow \sim p$

(iii) $[(p \vee q) \wedge \sim p] \Rightarrow q$

p	q	$\sim p$	$\sim q$	$p \vee q$	$p \rightarrow q$
0	0	1	1	0	1
0	1	1	0	1	1
1	0	0	1	1	0
1	1	0	0	1	1

(i) From the table, we find that when both p and $p \rightarrow q$ are true then q is true. This proves
$$[p \wedge (p \rightarrow q)] \Rightarrow q$$

(ii) From the table, we find that when both $p \rightarrow q$ and $\sim q$ are true, then $\sim p$ is true. This proves $[(p \rightarrow q) \wedge \sim q] \Rightarrow \sim p$.

(iii) From the table, we find that when both $p \vee q$ & $\sim p$ are true. then q is true. This proves
$$[(p \vee q) \wedge \sim p] \Rightarrow q.$$

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Rules of Inference :-

Consider a set of propositions $p_1, p_2, p_3 --- p_n$ and q. Then a compound proposition of the form $[p_1 \wedge p_2 \wedge p_3 \wedge .... \wedge p_n] \rightarrow q$ is called an Argument.

Here $p_1, p_2, ... p_n$ are called the premises hypothesis of the argument and q is called the conclusion of the argument.

This argument is represented in a tabular form

$$\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \vdots \\ p_n \\ \hline \therefore Q \end{array}$$

The argument is said to be valid if whenever each of the premises $p_1, p_2, --- p_n$ is true then the conclusion Q is true.

OR in other words if $(p_1 \wedge p_2 ...., \wedge p_n) \rightarrow Q$ is valid when $(p_1 \wedge p_2 --- \wedge p_n) \Rightarrow Q$

In the argument, the premises are always considered to be true (and hence the name hypothesis), whereas the conclusion may be true or false.

The conclusion is true only in the case of a valid argument.

We use the rules of inference to establish the validity of the arguments.

Rules of Inference

Rule of Inference	Logical Implication	Name of the Rule.
1) p q $\therefore p \wedge q$	–	Rule of conjuction
2) $p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Rule of conjuctive simplification.
3) p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Rule of disjunctive amplification.
4) p $p \rightarrow q$ $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Rule of detachment or modus pones.
5) $p \rightarrow q$ $\sim q$ $\therefore \sim p$	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$	Modus Tollens.
6) $p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Law of syllogism
7) $p \vee q$ $\sim p$ $\therefore q$	$[(p \vee q) \wedge \sim p] \rightarrow q$	Rule of disjunctive syllogism.
8) $\sim p \rightarrow F_0$ $\therefore p$	$(\sim p \rightarrow F_0) \rightarrow p$	Rule of contradiction.

Problems:-

1. Test whether the following is a valid argument.

If sachin hits a century, then he gets a free car
Sachin hits a century.

$\therefore$ sachin gets a free car

Sol^n: Let p: Sachin hits a century
q: Sachin gets a free car

Then, the given statement reads

$$\begin{array}{r} p \to q \\ p \\ \hline \therefore q \end{array}$$

In view of Modus pones, this is a valid argument

2. Test,

If sachin hits a century, he gets a free car.
Sachin does not get a free car

$\therefore$ Sachin has not hit a century.

Sol^n. Let p: sachin hits a century.
q: sachin gets a free car.

Then the given argument reads

$$\begin{array}{r} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

In view of Modus Tollen rule, the argument is valid.

3. Test,

If sachin hits a century, he gets a free car
sachin gets a free car

$\therefore$ Sachin has hit a century.

Rules of Inference

Rule of Inference	Logical Implication	Name of the Rule.
1) $\dfrac{p \quad q}{\therefore p \wedge q}$	-	Rule of conjuction
2) $\dfrac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Rule of conjuctive simplification.
3) $\dfrac{p}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$	Rule of disjunctive amplification.
4) $\dfrac{p \quad p \rightarrow q}{\therefore q}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Rule of detachment or modus pones.
5) $\dfrac{p \rightarrow q \quad \sim q}{\therefore \sim p}$	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$	Modus Tollens.
6) $\dfrac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Law of syllogism
7) $\dfrac{p \vee q \quad \sim p}{\therefore q}$	$[(p \vee q) \wedge \sim p] \rightarrow q$	Rule of disjunctive syllogism.
8) $\dfrac{\sim p \rightarrow F_0}{\therefore p}$	$(\sim p \rightarrow F_0) \rightarrow p$	Rule of contradiction.

Sol^n: Let p: Sachin hits a century
q: Sachin gets a free car
Then, the given argument reads

$$\begin{array}{l} p \to q \\ \quad q \\ \hline \therefore\ p \end{array}$$

We note that $p \to q$ & q are true, there is no rule which asserts that p must be true. Indeed, p can be false when $p \to q$ & q are true.

p	q	$p \to q$	$(p \to q) \wedge q$
0	1	1	1

Thus $[(p \to q) \wedge q] \to p$ is not a tautology.
$\therefore$ the given argument is not a valid.

4. Test,

If I drive to work, then I will arrive tired
I am not tired (when I arrive at work)

$\therefore$ I don't drive to work

Sol^n: Let p: I drive to work
q: I arrive tired
Then the given argument reads

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

In view of Modus Tollens rule, this is valid.

5. Test,

I will become famous or I will not become a musician
I will become a musician

$\therefore$ I will become famous.

Sol^n: Let p: I will become famous

q: I will become a musician

Then, the given argument reads

$$\begin{array}{r} p \vee \sim q \\ q \\ \hline \therefore p \end{array}$$

This argument is logically equivalent to

$$\begin{array}{r} q \to p \\ q \\ \hline \therefore p \end{array}$$

In view of the Modus pones, Rule, this argument is valid.

6. Test,

If I study, then I do not fail in the exam^n

If I don't fail in the exam^n, my father gifts a two-wheeler to me

---

$\therefore$ If I study then my father gifts a two-wheeler to me.

Sol^n: Let p: I study

q: I don't fail in the examination

r: My father gifts a two-wheeler to me.

Then the given argument reads

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

In view of Rule of syllogism, this is a valid argument.

7. Test,

If Ravi goes out with friends, he will not study

If Ravi does't study, his father become angry

His father is not angry

$\therefore$ Ravi has not gone out with friends.

Sol? Let p: Ravi goes out with friends

q: Ravi does't study.

r: His father gets angry

Then the argument reads

$$\begin{array}{l} p \to q \\ q \to r \\ \sim r \\ \hline \therefore \sim p \end{array}$$

This argument is logically equivalent to

$$\begin{array}{l} p \to r \\ \sim r \\ \hline \therefore \sim p \end{array}$$

In view of Modus Tollens rule, this is a valid.

8. Test whether the following arguments are valid

(i)
$$\begin{array}{l} p \to q \\ r \to s \\ p \vee r \\ \hline \therefore q \vee s \end{array}$$

(ii)
$$\begin{array}{l} p \to q \\ r \to s \\ \sim q \vee \sim s \\ \hline \therefore \sim (p \wedge r) \end{array}$$

Sol? (i) We note that

$(p \to q) \wedge (r \to s) \wedge (p \vee r) \Leftrightarrow (p \to q) \wedge (r \to s) \wedge (\sim p \to r)$

$\Leftrightarrow (p \to q) \wedge (\sim p \to r) \wedge (r \to s)$ (comm)

$\Leftrightarrow (p \to q) \wedge (\sim p \to s)$ (Rule of syllogism)

$\Leftrightarrow (\sim q \to \sim p) \wedge (\sim p \to s)$ (using contrative)

$\Leftrightarrow \sim q \to s$, (Rule of Syllogism)

$\Leftrightarrow q \vee s$

(x) $(p\to q)\wedge(r\to s)\wedge(\sim q\vee\sim s)$

$\Leftrightarrow (p\to q)\wedge(r\to s)\wedge(q\to\sim s)$

$\Leftrightarrow (p\to\sim s)\wedge(r\to s)$ (by comm & Syllogism)

$\Leftrightarrow (p\to\sim s)\wedge(\sim s\to\sim r)$, (by Contrative)

$\Rightarrow p\to\sim r$ (Rule of syllogism)

$\Rightarrow \sim p\vee\sim r \Leftrightarrow \sim(p\wedge r)$

This shows that the given argument is valid.

9. prove the validity of the following arguments:

(i)
$$\begin{array}{l} p\to r \\ \sim p\to q \\ q\to s \\ \hline \therefore \sim r\to s \end{array}$$

(ii)
$$\begin{array}{l} (\sim p\vee\sim q)\to(r\wedge s) \\ r\to t \\ \sim t \\ \hline \therefore p \end{array}$$

Sol^n: (i) We note that

$(p\to r)\wedge(\sim p\to q)\wedge(q\to s)$

$\Leftrightarrow (p\to r)\wedge(\sim p\to s)$ (by Rule of syllogism)

$\Leftrightarrow (\sim r\to\sim p)\wedge(\sim p\to s)$ (by contrative)

$\Leftrightarrow \sim r\to s$, (Rule of syllogism)

This shows that the given argument is valid.

(ii) We note that

$[(\sim p\vee\sim q)\to(r\wedge s)]\wedge(r\to t)\wedge(\sim t)$

$\Leftrightarrow [(\sim p\vee\sim q)\to(r\wedge s)]\wedge\sim r$ (Modus Tollens)

$\Leftrightarrow [(\sim p\vee\sim q)\to(r\wedge s)]\wedge(\sim r\vee\sim s)$ (Rule of disjunctive amplification)

$\Leftrightarrow [(\sim p\vee\sim q)\to(r\wedge s)]\wedge[\sim(r\wedge s)]$ (by DeMorgans)

$\Rightarrow \sim(\sim p\vee\sim q)$, (by Modus Tollens)

$\Leftrightarrow p\wedge q$ by DeMorgans

$\Rightarrow p$, (Rule of conjunctive simplification)

10. Test the validity of the following argument:

$$\begin{array}{l} p \to q \\ q \to s \\ r \to \sim s \\ \sim p \veebar r \\ \hline \therefore \sim p \end{array}$$

Sol: We note that

$$(p \to q) \wedge (q \to s) \wedge (r \to \sim s) \wedge (\sim p \veebar r)$$
$$\Rightarrow (p \to s) \wedge (s \to \sim r) \wedge (\sim p \veebar r)$$
$$\Rightarrow (p \to \sim r) \wedge (\sim p \veebar r)$$
$$\Leftrightarrow (r \to \sim p) \wedge (r \veebar \sim p).$$

Now, $r \veebar \sim p$ is true only in the following two possible cases:

(a) r is true and $\sim p$ is false

(b) r is false & $\sim p$ is true.

In case (a), $r \to \sim p$ is false & in case (b), $r \to \sim p$ is true. Hence it is only in case (b) that the RHS of (i) remains true. Thus $\sim p$ is true is a valid conclusion. This means that the given argument is valid.

Assignments :-

1. Test the validity of the following argument

If I study, I will not fail in the examination

If I do not watch TV in the evenings, I will study

I failed in the examination

∴ I must have watched TV in the evenings.

2. Consider the following argument:

I will get grade A in this course or I will not graduate

If I do not graduate, I will join the army

∴ I will not join the army.

Is this a valid argument ?

3. Test the validity of the following arguments:

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(i) $p \wedge q$

$p \to (q \to r)$

$\therefore r$

(ii) p

$p \to \sim q$

$\sim q \to \sim r$

$\therefore \sim r$

(iii) $p \to r$

$q \to r$

$\therefore (p \vee q) \to r$

4. Prove that the following are valid arguments:

(i) $p \to (q \to r)$

$\sim q \to \sim p$

p

$\therefore r$

(ii) $\sim p \leftrightarrow q$

$q \to r$

$\sim r$

$\therefore p$.

5. Prove that the following are valid arguments:

(i) $p \to (q \to r)$

$p \vee \sim s$

q

$\therefore s \to r$

(ii) $p \to (q \wedge r)$

$r \to s$

$\sim (q \wedge s)$

$\therefore \sim p$

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Open Statements :-

We present a few statements involving variables x, y, z etc.

(i) $p(x) : x \leq 5$

(ii) $q(x,y) : x+5 = y$

(iii) $r(x,y,z) : x^2+y^2 = z^2$

Statement of this kind are called open statements. These open statements turn out to be propositions for specific value of the variables which is either True (T) or False (F).

Ex: for (i), (ii) & (iii)

(i) $p(1) : 1 < 5$, $p(2) : 2 < 5$, $p(3) : 3 < 5$, $p(4) : 4 < 5$

are all True statements

$p(6) : 6 < 5$, $p(7) : 7 < 5$ ... are False statements.

(ii) $q(1,6) : 1+5 = 6$ ~~are~~ is True

$q(1,7) : 1+5 \neq 7$ is False

(iii) $r(3,4,5) : 9+16 = 25$ is true

$r(1,2,3) : 1+4 \neq 9$ is False.

Definition :- It is a statement which involves one or more variable which can be either true or false

Note :- $p(x)$, $q(x,y)$, $r(x,y,z)$ are called predicates.

Quantifiers :- open sentences involving words of the form "for all" or "for some" with reference to the variables involved are called Quantifiers.

Universal quantifiers:- Sentences involved with words of the type for all / for every symbolically $\forall$ with reference to the variables are called Universal quantifiers.

Existential quantifiers:- Sentences involved with words of the type for some / there exists, with Symbolically $\exists$ are called Existential quantifiers.

Quantified Statement:- Any statement involved with either of these two quantifiers is called a quantified Statement.

The truth value of a quantified statement (involved with $\forall / \exists$) is,

(i) $\forall x\,[p(x)]$ is true (T) only when $p(x)$ is true for every x belonging to a set S.

(ii) $\exists x\,[p(x)]$ is False (F) only when $p(x)$ is false for every x belonging to a set S.

The Negation of a quantified statement is

$$\sim\{\forall x[p(x)]\} \equiv \exists x, [\sim p(x)]$$

$$\sim\{\exists x[p(x)]\} \equiv \forall x, [\sim p(x)]$$

Problems:-

1. If $A = \{1, 2, 3, 4, 5\}$ is the Universal set, determine the truth values of each of the following statements.

(i) $(\forall x \in A)(x+2 < 10)$ (ii) $(\exists x \in A)(x+2 = 10)$

(iii) $(\forall x \in A)(x^2 \leq 25)$ (iv) $(\exists x \in A)(x^2 - 5x + 6 = 0)$

Soln: (i) Let $p_1(x) : x+2 < 10$

$p_1(1) : 3 < 10$, $p_1(2) : 4 < 10$ $p_1(3) = 5 < 10$

$p_1(4) : 6 < 10$, $p_1(5) : 7 < 10$

These are all true.

$\therefore p_1(x)$ is true for every $x \in A$.

Thus the truth value of $p_1(x)$ is True (T)

(ii) $p_2(x) : x+2 = 10$.

$p_1(1) : 1+2 = 3 \neq 10$

$p_1(2) : 2+2 = 4 \neq 10$

$p_1(3) : 3+2 = 5 \neq 10$

$p_1(4) : 4+2 = 6 \neq 10$

$p_1(5) : 5+2 = 7 \neq 10$

We observe That none of them is equal to RHS.

$\therefore p_2(x)$ is false for every $x \in A$.

Thus the truth value of $p_2(x)$ is False (F).

(iii) Let $p_3(x) : x^2 \leq 25$

$p_1(1) : 1 \leq 25$ $p_4(4) : 16 \leq 25$

$p_2(2) : 4 \leq 25$ $p_5(5) : 25 \leq 25$

$p_3(3) : 9 \leq 25$

$\therefore p_3(x)$ is true for every $x \in A$.

Thus the truth value of $p_3(x)$ is true

(iv) Let $p_4(x) : x^2 - 5x + 6 = 0$

$p_4(1) : 1 - 5 + 6 = 7 - 5 = 2 \neq 0$

$p_4(2) : 4 - 10 + 6 = 10 - 10 = 0$

$p_4(3) : 9 - 15 + 6 = 15 - 15 = 0$

$p_4(4) : 16 - 20 + 6 = 22 - 20 \neq 0$

$p_5(5) : 25 - 25 + 6 = 6 \neq 0$

We observe that $p_4(x)$ is satisfied for some values of $x = 2, 3$. These belongs to A.

Thus the truth value of $p_4(x)$ is true (T).

2. What is the truth value of $\forall x, p(x)$ and $\exists x, p(x)$ where $p(x)$ is the statement (i) $x^2 < 10$ (ii) $x^2 > 10$ and the universal set consists of positive integers not exceeding 4.

Soln: Let $S = \{1, 2, 3, 4\}$

(i) $p_1(x) : x^2 < 10$

$p_1(1) : 1 < 10$

$p_1(2) : 4 < 10$

$p_1(3) : 9 < 10$

$p_1(4) : 16 \not< 10$

Here $16 < 10$ is not true.

Thus the truth value of $[\forall x, p(x)]$ is False (F)

& $[\exists x, p(x)]$ is True (T)

(ii) $p_2(x) : x^2 > 10$

$p_2(1) : 1 \not> 10$

$p_2(2) : 4 \not> 10$

$p_2(3) : 9 \not> 10$

$p_2(4) : 16 > 10$

Here $1, 4, 9 > 10$ is False & $16 > 10$ is true

Thus the truth value of $\forall x\, p(x)$ is False (F)

and $[\exists x, p(x)]$ is True (T).

Further Refere T.b.

3. Let 'S' be the set of all integers representing the universal set and let $p(x)$, $q(x)$, $r(x)$ be the open statements represented as follows.

$p(x): x+2<10$, $q(x): x^2 \le 25$, $r(x): x>5$

Write down the truth values of the following.

(i) $p(4)$ (ii) $p(5) \vee \sim r(3)$ (iii) $\sim p(3) \wedge \sim q(4)$

(iv) $p(4) \rightarrow q(2) \wedge r(3)$

Sol:

Sol^n: (i) $p(x): x+2<10$

$p(4): 6<10$ — True.

(ii) $p(x): x+2<10$ $\quad$ $q(x): x^2 \geq 25$

$p(5): 7<10$ — True $\quad$ $r(3): 3>5$ — False

$p(5)$	$r(3)$	$\sim r(3)$	$p(5) \vee \sim r(3)$
T	F	T	T

$\therefore$ $p(5) \vee \sim r(3)$ is true

(iii) $p(x): x+2<10$ $\quad$ $q(x): x^2 \le 25$

$p(3): 5<10$ — true $\quad$ $q(4): 16 \le 25$ — True

$p(3)$	$q(4)$	$\sim p(3)$	$\sim q(4)$	$\sim p(3) \wedge \sim q(4)$
T	T	F	F	F

$\therefore$ $\sim p(3) \wedge \sim q(4)$ is False

(iv) $p(x): x+2<10$ $\quad$ $q(x): x^2 \le 25$ $\quad$ $r(x): x>5$

$p(4): 6<10$ — True $\quad$ $q(2): 4 \le 25$ False $\quad$ $r(3): 3>5$ False

$p(4)$	$q(2)$	$r(3)$	$q(2) \wedge r(3)$	$p(4) \rightarrow q(2) \wedge r(3)$
T	F	F	F	F

$\therefore$ $p(4) \rightarrow q(2) \wedge r(3)$ is False.

VIMP 4. Write the following proposition in the symbolic form and find its negation.

"If all triangles are right angled then no triangle is equiangular." (July 2005, 07)

Sol^n: Let S be the universal set consisting of all the Δles

Let $p(x)$: x is a right angled

$q(x)$: x is an equiangular triangle

Symbolic form : $[\forall x\, p(x)] \rightarrow [\forall x \sim q(x)]$

$$P \rightarrow Q$$

$$\cancel{\sim(P \rightarrow Q) \Rightarrow \sim[P \vee \sim Q]}$$
$$\cancel{\Rightarrow \sim P \wedge Q}$$

$$\sim(p \rightarrow q) \Rightarrow p \wedge \sim q$$

$$\Rightarrow$$

$$\sim(P \rightarrow Q) \Rightarrow P \wedge \sim Q$$

$$\Rightarrow \{\forall x\, p(x)\} \wedge [\exists x \sim q(x)]$$

All triangles are right angled and there are some triangles which are equiangular.

5. Write down the following proposition in symbolic form and find its negation

"For all integers n, if n is not divisible by 2 then n is odd".

Sol^n: Let Z be the set of ~~inte~~ all integers be the universal set.

Let $p(n)$: n is divisible by 2

$q(n)$: n is odd.

Symbolic form : $\forall n, [\sim p(n) \rightarrow q(n)]$

Negation is, $\sim\{\forall n\, [\sim p(n) \rightarrow q(n)]\}$

$$\Rightarrow \exists n \sim [\sim p(n) \rightarrow q(n)]$$

$\Rightarrow \exists n, \sim[\sim p \vee q(n)]$ $\qquad p \to q \Rightarrow$ ~~$p \vee q$~~ $\sim p \vee q$

$\Rightarrow \exists n \;\; p(n) \wedge \sim q(n)$

$\exists n \; [p(n) \wedge \sim q(n)]$

There exists some integers n where n is not divisible by 2 and n is not odd.

V Imp

6. Write the negation of the follwing statements. Z is the Universal set
(i) Some rectangles are square
(ii) All youngsters like Cricket and some elders like foot ball.
(iii) For all $x \in Z$, $2x^2 - 3x + 1 \neq 0$
(iv) If k, m, n are integers where $(k-m)$ & $(m-n)$ are odd then $(k-n)$ is even. (Jan 2008, 10)

Sol^n: (i) Let 'S' be the set of all rectangle

$p(x)$: Rectangles are square.

Symbolic form: $\{\exists x \in S \, (p(x))\}$

Negation is, $\sim\{\exists x \in S \, (p(x))\}$

$\Rightarrow \forall x \in S \;\; \sim p(x)$

"Every rectangle is not a square."

(ii) Let 'S' be the set of all people.

$p(x)$: Youngsters like Cricket

$q(x)$: Elder's Like foot ball.

Symbolic form $[\forall x, p(x)] \wedge [\exists x, q(x)]$

Negation is, $\sim\{[\forall x, p(x)] \wedge [\exists x \; q(x)]\}$

$\Rightarrow [\exists x \sim p(x)] \vee [\forall x \sim q(x)]$

"Some youngsters don't like cricket and All elders don't like foot ball."

(III) Let $p(x): 2x^2-3x+1 \neq 0$, Z is the set of all integers

Symbolic form: $(\forall x \in Z)\ (2x^2-3x+1 \neq 0)$

Negation is, $\sim\{(\forall x \in Z)\ (2x^2-3x+1 \neq 0)\}$

$\Rightarrow \exists x \in Z\ (2x^2-3x+1 = 0)$

For some integer, $2x^2-3x+1=0$.

(IV) Let $P = p(k,m)$: $(k-m)$ is odd

$Q = q(m,n)$: $(m-n)$ is odd

$R = r(k,n)$; $(k-n)$ is even

Also, Z is the set of all integers

Symbolic: $(\forall\ k,m,n \in Z)[(P \wedge Q) \rightarrow R]$

Negation is, $\sim\{(\forall\ k,m,n \in Z)\ ((P \wedge Q) \rightarrow R)\}$

$\Rightarrow \exists\ k,m,n \in Z\ \sim\{(P \wedge Q) \rightarrow R\}$

$\Rightarrow \exists\ k,m,n \in Z\ \sim\{\sim(P \wedge Q) \vee R\}$

$\Rightarrow \exists\ k,m,n \in Z\ (P \wedge Q) \wedge \sim R$

"There exists integers k, m, n such that both $(k-m)$, $(m-n)$ are odd and $(k-n)$ is not even."

V. Imp

7. For the Universe of all integers, define the following open statements. $p(x): x>0$, $q(x)$: x is even, $r(x)$: x is a perfect square, $s(x)$: x is divisible by 3, $t(x)$: x is divisible by 7. Write down the foll. quantified statements in symbolic form.

(i) Atleast one integer is even

(ii) Some even integers are divisible by 3

(iii) Every integer is either even or odd.

(iv) If x is even and a perfect square then x is not divisible by 3.

(v) If x is odd or not divisible by 7 then x is divisible by 3. (Jan 2010)

Soln: Z is the set of integers is the Universal set

Symbolic form is

(i) $\exists x \in Z \; q(x)$

(ii) $\exists x \in Z \; [q(x) \wedge s(x)]$

(iii) $\exists x \in Z \; [q(x) \underline{\vee} \sim q(x)]$

(iv) $\forall x \in Z \; [\{q(x) \wedge r(x)\} \rightarrow \sim s(x)]$

(v) $\forall x \in Z \; [\{\sim q(x) \vee \sim t(x)\} \rightarrow s(x)]$

VImp

8. Negate and simplify each of the following

(i) $\forall x, [p(x) \wedge \sim q(x)]$

(ii) $\exists x, [p(x) \vee q(x) \longrightarrow r(x)]$ (June 2010)

Soln. (i) $\sim \{\forall x, [p(x) \wedge \sim q(x)]\}$

$= \exists x, \sim [p(x) \wedge \sim q(x)]$

$= \exists x, [\sim p(x) \vee q(x)]$

$= \exists x, [p(x) \rightarrow q(x)]$ is the req. negation

(ii) $\sim [\exists x \{p(x) \vee q(x) \rightarrow r(x)\}]$

$= \forall x, \sim [p(x) \vee q(x) \rightarrow r(x)]$

$= \forall x, \sim [\sim \{p(x) \vee q(x)\} \vee r(x)]$

$= \forall x, [p(x) \vee q(x)] \wedge \sim r(x)$ is the req.

VImp

9. Write the following proposition in symbolic form and find its negation.

"All integers are rational numbers and some rational numbers are not integers." [Dec 2010]

Soln: Let $p(x)$: x is an integer

$q(x)$: x is a rational number

Also, Z is set of all integer

Q is set of all rational number.

Symbolic form. $[\forall x \in Z, q(x)] \wedge [\exists x \in Q, \sim p(x)]$

Negation is, $\sim \{[\forall x \in Z, q(x)] \wedge [\exists x \in Q, \sim p(x)]\}$

$$= [\exists x \in Z \ \sim q(x)] \vee [\forall x \in Q \ \ p(x)]$$

"There exists some integers which are not rational numbers or all rational numbers are integers."

VIMP

10. For the following statement, State the converse, Inverse and Contrapositive. The universe consists of all integers.

"If m divides n and n divides p, then m divides p."

(June 2011)

Sol^n. $P = p(m,n)$: m divides n

$Q = q(n,p)$: n divides p

$R = r(m,p)$: m divides p

Symbolic form: $(P \wedge Q) \longrightarrow R$

(i) Converse ($p \to q$ is $q \to p$)

$(P \wedge Q) \to R$ is $R \to (P \wedge Q)$

i.e "If m divides p then m divides n & n divides p"

(ii) Inverse ($p \to q$ is $\sim p \to \sim q$)

$(P \wedge Q) \to R$ is $\sim (P \wedge Q) \longrightarrow \sim R$

i.e "If m does not divide n or n does not divide p then m doesn't divide p."

(iii) Contrapositive ($p \to q$ is $\sim q \to \sim p$)

$(P \wedge Q) \to R$ is $\sim R \to \sim (P \wedge Q)$

$\Rightarrow \sim R \to (\sim P \vee \sim Q)$

"If m does not divide p then m doesn't divide n or n doesn't divide p."

√ Imp 11. Prove the following argument is valid where 'c' is the specification element of the Universe S.

$$\forall x, [p(x) \to q(x)]$$ [Jan 2010]
$$\forall x, [q(x) \to r(x)]$$
$$\sim r(c)$$
$$\overline{\therefore \sim p(c)}$$

Sol^n: we have $\forall x, [p(x) \to q(x)] \Rightarrow \forall c\ [p(c) \to q(c)]$ — ①

$\forall x, [q(x) \to r(x)] \Rightarrow \forall c, [q(c) \to r(c)]$ — ②

combine ① & ②

$[p(c) \to q(c)] \wedge [q(c) \to r(c)] \Rightarrow p(c) \to r(c)$ (by syllogism)

$[p(c) \to r(c)] \wedge \sim r(c) \Rightarrow \sim p(c)$ (Modus tollens)

Hence the argument is valid.

√ Imp 12. Establish the validity of the foll. argument [June 2010, Jan 17]

$$\forall x, [p(x) \vee q(x)]$$
$$\exists x, \sim p(x)$$
$$\forall x, [\sim q(x) \vee r(x)]$$
$$\forall x, [s(x) \to \sim r(x)]$$
$$\overline{\therefore \exists x, \sim s(x)}$$

Sol^n. we have $\forall x, [p(x) \vee q(x)] \Rightarrow [p(a) \vee q(a)]$ — ①

$\exists x, \sim p(x) \Rightarrow \sim p(a)$ — ②

① & ② $\Rightarrow [p(a) \vee q(a)] \wedge \sim p(a) \equiv q(a)$ — ③.

$\forall x, [\sim q(x) \vee r(x)] \Rightarrow \sim q(a) \vee r(a)$ — ④

③ & ④ $q(a) \wedge [\sim q(a) \vee r(a)] \equiv$

$\Rightarrow q(a) \wedge [q(a) \to r(a)] \Rightarrow r(a)$ — ⑤

Further, $\forall x, [s(x) \to \sim r(x)]$

$\Rightarrow s(a) \to \sim r(a)$ — (6)

$p \to q \equiv \sim q \to \sim p$ for (6).

$s(a) \to \sim r(a) \Rightarrow \sim(\sim(r(a))) \to \sim s(a)$

$\Rightarrow r(a) \to \sim s(a)$ — (7)

combine (5) & (7)

$r(a) \wedge [r(a) \to \sim s(a)] \Rightarrow \sim s(a)$

$\Rightarrow \exists x, \sim s(x)$

Hence the argument is valid.

Definitions and proofs of Theorems

Definition can be understood as a statement of the meaning of a word.

We present all the three methods of proof in logical language using the rules of inference and laws of logic.

Direct proof :- We assume p is true (T) and show that q is true (T). Therefore we conclude that $p \to q$ is true (T).

Indirect proof :- We assume that $\sim q$ is true which is equivalent to, p is true (T) and q is false (F). This results in we establish $\sim p$ is true.

$\sim q \to \sim p$ (contrapositive) being true.

W.K.T $p \to q \Leftrightarrow \sim q \to \sim p$, we conclude that $p \to q$ is true (T).

Proof by contradiction :-

We assume $p \to q$ is False (F) which is equivalent to p is true (T) and q is false (F). Here we start with q is false (F) and establish that p is false (F). Since p is already true (T) we arrive at a contradiction. Therefore we conclude that $p \to q$ is true (T).

Disproof of a given proposition

If there exist atleast one element $x_0 \in S$ such that $p(x_0)$ is false then the truth value of the given proposition is false. This is equivalent to providing a counter example disproving the given proposition.

Further, if the set 'S' consists of few number of elements, truthfulness of the proposition $[\forall x \in S, p(x)]$ can be justified verifying that $p(x)$ is true for every x belonging to the given S.

This method is called Method of Exhaustion.

Problems:-

1. Give (i) A direct proof (ii) An indirect proof (iii) A proof by contradiction for the following statement "If n is an even integer then n+3 is an odd integer."

<u>Sol</u>: Let p: n is an even integer

q: n+3 is an odd integer

(i) <u>Direct proof</u>: n is even implies that $n = 2r,\ r \in Z$

$\therefore\ n+3 = 2r+3 = 2r+2+1$

$= 2(r+1)+1$

$= 2k+1 \quad k \in Z$, which is odd

Thus if n is even then $n+3$ is odd.

(ii) <u>Indirect proof</u>: We assume that $\sim q$ is true or p is True & q is false.

i.e p: n is an even integer

q: n+3 is an even integer

$\therefore\ n+3 = 2r$ or $n = 2r-3 = \cancel{2r-2}-1$

$= 2(r-2)+1$

$= 2m+1,\ m \in Z$

That is to say that n is an odd integer which is equivalent to $\sim p$ being true.

$\therefore \sim q \to \sim p$ is true. But $\sim q \to \sim p \Leftrightarrow p \to q$

Hence $p \to q$ is true.

Proof by contradiction :

Assume $p \to q$ is false or p is true & q is false.

i.e p: n is an even integer

q: $n+3$ is an even integer

$\therefore n+3 = 2r$ or $n = 2r-3$

$= 2(r-2)+1 = 2m+1, \; m \in Z$

That is to say that n is an odd integer which contradicts the proposition p that n is an even integer. Hence $p \to q$ is true.

Thus if n is even then $n+3$ is odd.

VIMP

2. Give a (i) A direct proof (ii) An indirect proof (iii) A proof by contradiction for the following statement.

"If n is an odd integer, then $(n+9)$ is an even integer." (Dec 2010 Jan 2017)

Sol^n: Let p: n is an odd integer

q: $n+9$ is an even integer.

(i) Direct Proof: n is an odd integer $\Rightarrow n = 2r+1, r \in Z$

$\therefore n+9 = (2r+1)+9 = 2r+10$

$= 2(r+5) = 2m$ (say), $r+5 \in Z$

$\therefore n+9$ is an even integer.

Thus, if n is odd, then $n+9$ is even.

(ii) Indirect Proof : Assume that $\sim q$ is true or p is true & q is false.

p: n is an odd integer

q: $n+9$ is an odd integer.

$\therefore n+9 = 2r+1$

$\Rightarrow n = 2r+1-9 = 2r-8 = 2(r-4)$

$= 2m$

$\therefore n = 2m \quad , \; r-4 \in Z$

That is $n = 2m \Rightarrow$ 'n' is even. $\Rightarrow \sim p$ is true.

$\therefore \sim q \to \sim p$ is true

we have $\sim q \to \sim p \Leftrightarrow p \to q$, hence $p \to q$ is true

Thus if n is odd then $n+9$ is even.

(iii) <u>proof by contradiction</u>:

Assume $p \to q$ is false <u>or</u>

p is true & q is false.

p: n is an odd integer

q: $(n+9)$ is an odd integer

$\therefore n+9 = 2r+1$

$n = 2r+1-9 = 2r-8 = 2(r-4) = 2m$, $m \in Z$

That is to say that n is an even integer which contradicts the proposition p that n is an odd integer. Hence $p \to q$ is true.

Thus if n is odd then $(n+9)$ is even.

3. Establish that the sum of seven consecutive numbers is divisible by 7.

Sol^n: We shall use the direct method

If $x \in Z$, then the seven consecutive numbers are

$x = x_0$, $x_1 = x_0+1$, $x_2 = x_0+2$, $x_3 = x_0+3$, $x_4 = x_0+4$

$x_5 = x_0+5$, $x_6 = x_0+6$

$$\therefore \sum_{i=0}^{6} x_i = x_0 + (x_0+1) + (x_0+2) + (x_0+3) + (x_0+4) + (x_0+5) + (x_0+6)$$

$$= 7x_0 + 21 = 7(x_0+3) = 7r, \ r \in Z$$

That is

$$\sum_{i=0}^{6} x_i = 7r \Rightarrow \sum_{i=0}^{6} x_i \text{ is divisible by } 7.$$

Hence proved.

4. Show that for all odd integers

(i) The sum of two odd integers is even

(ii) The product of two odd integers is odd

<u>Sol</u>: Let 'S' be the set of all odd integers.

(i) Let $x, y \in S$

$\therefore x = 2m+1$ and $y = 2n+1 \quad m, n \in Z$

Now $x+y = (2m+1) + (2n+1)$

$= 2m+2n+2$

$= 2(m+n+1)$

That is, $x+y = 2r, \; r \in Z \Rightarrow x+y$ is even.

(ii) ~~x+y~~ $xy = (2m+1)(2n+1)$

$= 4mn + 2m + 2n + 1$

$= 2(mn + m + n) + 1$

$= 2r+1, \quad r \in Z$

$\therefore xy = 2r+1 \Rightarrow xy$ is odd

5. Given that $x \in Z$ and $(x-1)$ is divisible by 5, prove by direct method ~~that~~ x^2+4 is divisible by 5.

Sol: By data, $(x-1) = 5k \Rightarrow x = 5k+1 \quad k \in Z$

Now, $x^2+4 = (5k+1)^2 + 4$

$= 25k^2 + 10k + 1 + 4$

$= 25k^2 + 10k + 5$

$= 5(5k^2 + 2k + 1)$

$= 5r$

That is, $x^2+4 = 5r, \; r \in Z$

$\Rightarrow x^2+4$ is divisible by 5.

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Types of Quantifiers :-

1. **Universal Quantifiers :-** Denoted by $\forall x$ and is read as "for all x", "for any x", "for each x", "for every x" $\forall x, y$ – "for all x, y"

Ex: $\forall x \in S$, $p(x)$ – It is read as "for all x belong to S, where $p(x)$ is the open statement. The variable 'x' is called as a bound variable.

2) **Existential Quantifier :-** Denoted by $\exists x$ and is read as "for some x", "for atleast one x", "there exist on x"

Ex: $\exists x \in S$, $p(x)$ – "there exist x which belongs to S and $p(x)$ is an open statement.

Problems :-

1. For the universe of all integers, Let $p(x)$: $x > 0$, $q(x)$: x is even, $r(x)$: x is a perfect square, $S(x)$: x is divisible by 3, $r(x)$: x is divisible by 7.

write down the following quantified statement in symbolic form.

(i) Atleast one integer is even $\exists x, q(x)$

(ii) There exists a positive integer that is even
$\exists x, [p(x) \wedge q(x)]$

(iii) Some even integers are divisible by 3.
$\exists x, [q(x) \wedge S(x)]$

(iv) Every integer is either even or odd

$\forall x, [q(x) \veebar \sim q(x)]$

(v) If x is even and a perfect square, then x is not divisible by 3.

$\forall x, [\{q(x) \wedge r(x)\} \rightarrow \sim s(x)]$

(vi) If x is odd or is not divisible by 7, then x is divisible by 3.

$\forall x, [\{\sim q(x) \vee \sim t(x)\} \rightarrow s(x)]$

2. Consider the open statements $p(x)$, $q(x)$, $r(x)$, $s(x)$, $t(x)$ of problem ①. Express each of the following symbolic statements in words and indicate its truth value.

(i) $\forall x, [r(x) \rightarrow p(x)]$

(ii) $\exists x, [s(x) \wedge \sim q(x)]$

(iii) $\forall x, [\sim r(x)]$

(iv) $\forall x, [r(x) \vee t(x)]$

Sol? (i) For any integer x, if x is a perfect square, then $x>0$ — False (Take $x=0$)

(ii) For some integer x, x is divisible by 3 and x is not even. — True (Take $x=9$)

(iii) For any integer x, x is not a perfect square — False

(iv) For any integer x, x is a perfect square or x is divisible by 7 — False (take $x=8$)

Use of Quantifiers:-

Open statement:- It is a declarative statement which contains one or more variables.

It is not a statement, but when the variables in it are replaced by certain allowable choices, it can be called as a statement.

Ex:- i) $x+5=10$ (ii) $x^3<100$

open statements containing a variable denoted by $p(x)$, $q(x)$ etc.

Hence 'x' is called a free variable.

Ex: $p(x): x+5=10$, If $x=5$ then $p(5)=10$

∴ the truth value of $p(5)$ are

Negation	$\sim p(x)$
conjuction	$p(x) \wedge q(x)$
Disjunction	$p(x) \vee q(x)$
Conditional	$p(x) \rightarrow q(x)$
Biconditional	$p(x) \leftrightarrow q(x)$

1. Suppose the universe consists of all integers. Consider the following open statements:

$p(x): x \leq 5$ $\quad q(x): x+1$ is odd $\quad r(x): x>0$

Give the truth values of the following

(i) $p(2)$ (ii) $\sim q(4)$ (iii) $p(-1) \wedge q(1)$

(iv) $\sim p(3) \vee r(0)$ (v) $p(0) \rightarrow q(0)$

(vi) $p(1) \rightarrow q(2)$ (vii) $p(4) \vee [q(1) \wedge r(2)]$ (viii) $p(2) \wedge [q(0) \vee \sim r(2)]$

Sol^n: (i) $p(2): 2 \leq 3$ is true

(ii) $\sim q(4)$: $4+1 = 5$ is not odd - False

(iii) $p(-1) \wedge q(1)$

$p(-1): -1 \leq 3$ true

$q(1)$: $1+1 = 2$ is odd - false

$\therefore p(-1) \wedge q(1)$ is false

(iv) $\sim p(3) \vee r(0)$

$\sim p(3): 3 \nleq 3$ is false $\quad r(0): 0>0$ is false

$\therefore \sim p(3) \vee r(0)$ is false

(v) $p(0) \longrightarrow q(0)$

$p(0): 0 \leq 0$ true $\quad q(0): 0+1=1$ is odd

$\therefore p(0) \rightarrow q(0)$ is true

(vi) $p(1) \leftrightarrow \sim q(2)$

$p(1): 1 \leq 3$ true $\quad \sim q(2): 2+1=3$ is not odd false

$\therefore p(1) \leftrightarrow \sim q(2)$ is false

(vii) $p(4) \vee [q(1) \wedge r(2)] = 0 \vee [0 \wedge 1] = 0 \vee 0 = 0$

$p(4): 4 \leq 3$ false

$q(1): 1+1 = 2$ is odd false

$r(2): 2>0$ true

(viii) $p(2) \wedge [q(0) \vee \sim r(2)]$

$p(2): 2 \leq 3$ true $\quad q(0): 0+1=1$ is ~~odd~~ odd true

$\therefore p(2) \wedge [q(0) \vee \sim r(2)] =$ true.

3. Consider the following open statements with the set of all real numbers as the universe.

$p(x): |x| > 3$, $q(x): x > 3$

Find the truth value of the statement $\forall x, [p(x) \to q(x)]$ —— ①

Also, write down the converse, inverse and contrapositive of this statement and find their truth values.

Sol: We note that

$p(-4) \equiv |-4| > 3 \equiv 4 > 3$ is true

$q(-4) \equiv -4 > 3$ is false.

Thus $p(x) \to q(x)$ is false for $x = -4$.

Accordingly, the given statement ① is false.

The converse of the statement ① is $\forall x, [q(x) \to p(x)]$ —— ②

"For every real number x, if $x > 3$ then $|x| > 3$"

— True

The inverse of the statement ① is

$\forall x, [\sim p(x) \to \sim q(x)]$ —— ③

"For every real number x, if $|x| \le 3$, then $x \le 3$"

Statement ① & ② have the same truth values.

Thus ③ is a true statement.

The contrapositive of the statement ① is,

$\forall x, [\sim q(x) \to \sim p(x)]$ —— ④

"Every real number which is less than or equal to 3 has its magnitude less than or equal to 3."

statement ① & ④ have the same truth value.

Since ① is a false statement, so is its contra tve ④

VIMP. Consider the following open statements with the set of all real numbers as the universe.

$p(x): x \geq 0$, $q(x): x^2 \geq 0$, $r(x): x^2 - 3x - 4 = 0$

$s(x): x^2 - 3 > 0$

Determine the truth value of the following statements:

(i) $\exists x, p(x) \wedge q(x)$

Let $x = 2$, $p(2): 2 \geq 0$, $q(2): 4 \geq 0$ both are true

$\therefore \exists x\, [p(x) \wedge q(x)]$ is true.

(ii) $\forall x\, [q(x) \rightarrow q(x)]$

Let $x = -1$ $p(x): -1 \geq 0$ – False

$q(x): 1 \geq 0$ – true

$\therefore \forall x\, [p(x) \rightarrow q(x)]$ is true

(iii) $\forall x\, [q(x) \longrightarrow s(x)]$

Let $x = 1$, $q(1): 1 \geq 0$ – true

$s(1): 1 - 3 = -2 > 0$ is false

$\therefore \forall x\, [q(x) \rightarrow s(x)]$ is false

(iv) $\forall x\, [r(x) \vee s(x)]$

Let $x = 1$, $r(1): -7 = 0$ – False

$s(1): -2 > 0$ – False

$\forall x\, [r(x) \vee s(x)]$ is false

(v) $\exists x\, [p(x) \wedge r(x)]$

Let $x = 4$, $p(4): 4 > 0$ – true

$r(4): 4^2 - 3(4) - 4 = 0$ – true

$\therefore \exists x\, [p(x) \wedge r(x)]$ is true.

(vi) $\forall x\, [r(x) \longrightarrow p(x)]$

Let $x = -1$, $r(-1): 1 + 3 - 4 = 0$ – true

$p(-1): -1 > 0$ False

$\therefore \forall x\, [r(x) \rightarrow p(x)]$ is false.

5. Let $p(x): x^2-7x+10=0$, $q(x): x^2-2x-3=0$, $r(x): x<0$

Determine the truth or falsity of the following statements when the universe U contains only the integers 2 and 5. If a statement is false, provide a counter example or explanation.

(i) $\forall x,\ p(x) \rightarrow \sim r(x)$

(ii) $\forall x,\ q(x) \rightarrow r(x)$

(iii) $\exists x,\ q(x) \rightarrow r(x)$

(iv) $\exists x,\ p(x) \rightarrow r(x)$

Sol? Here the universe is $U=\{2,5\}$.

We note that $x^2-7x+10 \equiv (x-5)(x-2)$

$\therefore$ $p(x)$ is true for $x=5$ and 2.

That is $p(x)$ is true for all $x \in U$.

Further, $x^2-2x-3 \equiv (x-3)(x+1)$

$\therefore$ $q(x)$ is true only for $x=3$ and $x=-1$

Since $x=3$ and $x=-1$ are not in the universe.

$q(x)$ is false for all $x \in U$.

Obviously, $r(x)$ is false for all $x \in U$.

Accordingly:

(i) Since $p(x)$ is true for all $x \in U$ and $\sim r(x)$ is true for all $x \in U$, the statement $\forall x, p(x) \rightarrow \sim r(x)$ is true.

(ii) Since $q(x)$ is false for all $x \in U$ and $r(x)$ is false for all $x \in U$, the st $\forall x, q(x) \rightarrow r(x)$ is true

(iii) Since $q(x)$ and $r(x)$ are false for $x=2$,

the st. $\exists x, q(x) \rightarrow r(x)$ is true

(iv) Since $p(x)$ is true for all $x \in U$ but $r(x)$ is false for all $x \in U$, the Statement $p(x) \to r(x)$ is false for every $x \in U$. Consequently $\exists x, p(x) \to r(x)$ is false.

Rule of Negation:-

$$\sim [\forall x \; p(x)] \Leftrightarrow \exists x \, [\sim p(x)]$$

$$\sim [\exists x \; p(x)] \Leftrightarrow \forall x \, [\sim p(x)]$$

Ex:-/ problems:-

1. Find the Negation of the following statements when Z is the Universe.

$p(x)$: x is odd $\quad q(x)$: x^2-1 is even

$\forall x \, [p(x) \to q(x)]$

Sol? Negation is $\sim [\forall x \, \{p(x) \to q(x)\}]$

$\Leftrightarrow \exists x \, [\sim \{p(x) \to q(x)\}]$

$\Leftrightarrow \exists x \, [\sim \{\sim p(x) \vee q(x)\}]$

$\Leftrightarrow \exists x \, [p(x) \wedge \sim q(x)]$

Negation in words " There exist an x such that x is odd and x^2-1 is not even".

$\forall x \, [p(x) \to q(x)]$ is true

$\exists x \, [p(x) \wedge \sim q(x)]$ is false.

2. Negate and simplify each of the following

(i) $\exists x \, [p(x) \vee q(x)]$

$\Leftrightarrow \sim \{\exists x \, (p(x) \vee q(x))]\}$

$\Leftrightarrow \forall x \, [\sim \{p(x) \vee q(x)\}]$

$\Leftrightarrow \forall x \, [\sim p(x) \wedge \sim q(x)]$

(ii) $\forall x\, [p(x) \wedge \sim q(x)]$

$\Leftrightarrow \sim \forall x\, [p(x) \wedge \sim q(x)]$

$\Leftrightarrow \exists x\, [\sim \{p(x) \wedge \sim q(x)\}]$

$\Leftrightarrow \exists x\, [\sim p(x) \vee q(x)]$

(iii) $\forall x\, [p(x) \rightarrow q(x)]$

$\Leftrightarrow \sim \forall x\, [p(x) \rightarrow q(x)]$

$\Leftrightarrow \exists x\, [\sim \{p(x) \rightarrow q(x)\}]$

$\Leftrightarrow \exists x\, [\sim \{\sim p(x) \vee q(x)\}]$

$\Leftrightarrow \exists x\, [p(x) \wedge \sim q(x)]$

(iv) $\exists x\, [\{p(x) \wedge q(x)\} \rightarrow r(x)]$

$\Leftrightarrow \sim \exists x\, [\{p(x) \wedge q(x)\} \rightarrow r(x)]$

$\Leftrightarrow \forall x\, \sim [\{p(x) \wedge q(x)\} \rightarrow r(x)]$

$\Leftrightarrow \forall x\, \sim [\sim (p(x) \wedge q(x)) \vee r(x)]$

$\Leftrightarrow \forall x\, \sim (\sim (p(x) \wedge q(x)) \wedge \sim r(x)$

$\Leftrightarrow \forall x\, p(x) \wedge q(x) \wedge \sim r(x)$

3. Write the following proposition in symbolic form and find its Negation.

(i) "If all triangles are right angled then no Δles is equiangular"

Sol^n. Let $p(x)$: x is right angled Δle

$q(x)$: x is equi angular.

The given proposition is symbolically represented as

$\forall x\, p(x) \rightarrow \forall x \sim q(x)$

Negation is. $\sim [\forall x\, p(x) \rightarrow \forall x \sim q(x)]$

$\Leftrightarrow \sim [\sim \forall x\, p(x) \vee \forall x \sim q(x)]$

$\Leftrightarrow \forall x\, p(x) \wedge \exists x \sim [\sim q(x)]$

$\Leftrightarrow \forall x\, p(x) \wedge \exists x\, q(x)$

In words: All Δles are right angled and some Δles are equiangular.

(ii) "For all integers n, if n is not divisible by 2 then n is odd.

Sol^n: Let $p(x)$: n is divisible by 2

$q(x)$: n is odd

Given statement is $\forall x\, [\sim p(x) \longrightarrow q(x)]$

Negation is $\sim [\forall x\, \{\sim p(x) \rightarrow q(x)\}]$

$\Leftrightarrow \exists x \sim \{\sim p(x) \rightarrow q(x)\}$

$\Leftrightarrow \exists x \sim \{p(x) \vee q(x)\}$

$\Leftrightarrow \exists x\, [\sim p(x) \wedge \sim q(x)]$

$\therefore$ "For some integer n, n is not divisible by 2 and n is not odd".

Some important relations:-

$$\exists x\,[p(x) \wedge q(x)] \Rightarrow \exists x\, p(x) \wedge \exists x\, q(x)$$

$$\exists x\,[p(x) \vee q(x)] \Rightarrow \exists x\, p(x) \vee \exists x\, q(x)$$

$$\forall x\,[p(x) \wedge q(x)] \Leftrightarrow \forall x\, p(x) \wedge \forall x\, q(x)$$

$$\forall x\,[p(x) \vee q(x)] \Leftrightarrow \forall x\, p(x) \vee \forall x\, q(x)$$

The rule of universal specification:-

~~If an open statement $p(x)$ is proved to be true when x is replaced by an arbitary element 'a' from universe then the following quantified statement $\forall x\, p(x)$ is true.~~

If an open statement becomes true for all replacement by the numbers in a given universe then that open statement is true for any specified individual number in that universe.

i.e If $p(x)$ is an open statement for a given universe and if $\forall x\, p(x)$ is true ~~and~~ then $p(a)$ is true for each 'a' in the universe.

The Rule of universal Generalization:-

If an open statement $p(x)$ is proved to be true when x is replaced by an arbitary element 'a' from universe then the universally quantified statement $\forall x\, p(x)$ is true.

Problems :-

1. Verify whether the arguments are valid

(i) All mathematics professors ~~are~~ have studied calculus
Leona is a mathematics professor
∴ Leona has studied calculus.

Sol: Let $m(x)$: x is a mathematics professor
$c(x)$: x has studied calculus

Symbolically
$$\begin{array}{l} \forall x\,[m(x) \rightarrow c(x)] \\ m(l) \\ \hline \therefore c(l) \end{array}$$

$$\Leftrightarrow \quad \left.\begin{array}{l} m(l) \rightarrow c(l) \\ m(l) \\ \hline \therefore c(l) \end{array}\right\} \text{Universal specification \& Modus pones}$$

Steps	Reasons
1) $\forall x\,[m(x) \rightarrow c(x)]$	premise
2) $m(l) \rightarrow c(l)$	Step (1), Rule of specification premise.
3) $m(l)$	premise
4) $\therefore c(l)$	Step (2), (3) & Modus pones

∴ The given argument is valid.

(ii) In a Δle xyz there is no pair of angles of equal measure.
If a Δle has two sides of equal length then it is isosceles
If a Δle is isosceles then it has two angles of equal measure
If a Δle is isosceles then it has two sides of equal length.
∴ Δle xyz has no two sides of equal length

Sol^n. Let $p(t)$: Δle 't' has two sides of equal length

$q(t)$: t is an isosceles Δle

$r(t)$: t has two angles of equal measure

$\sim r(t)$.

$$\begin{array}{l} \forall t\,[p(t) \to q(t)] \\ \forall t\,[q(t) \to r(t)] \\ \hline \therefore \sim p(t) \end{array} \Leftrightarrow \begin{array}{l} \sim r(t) \\ p(t) \to q(t) \\ q(t) \to r(t) \end{array} \Leftrightarrow \begin{array}{l} \sim r(t) \\ p(t) \to r(t) \\ \therefore \sim p(t) \end{array}$$

– specification, syllgism, modus tollens

steps	Reason
1) $\forall t\,[p(t) \to q(t)]$	premise
2) $p(t) \to q(t)$	step (1) & Universal
3) $\forall t\,[q(t) \to r(t)]$	premise
4) $q(t) \to r(t)$	step (3) & V S
5) $p(t) \to r(t)$	(2) & (4), syllogism
6) $\sim r(t)$	premise
7) $\therefore \sim p(t)$	step (5) & (6), modus tollens

$\therefore$ The argument is valid.

(ii) No Engg student is bad in studies

Jeff is not bad in studies

$\therefore$ Jeff is an Engineering student

Sol^n. Let $p(x)$: x is Engineering student

$q(x)$: x is bad in studies

$$\begin{array}{l} \forall x\; p(x) \to \sim q(x) \\ \sim q(j) \\ \hline \therefore p(j) \end{array}$$

Steps	Reason
1) $\forall x\; p(x) \to \sim q(x)$	premise
2) $p(j) \to \sim q(j)$	(1) & V S
3) $\sim q(j)$	premise

Here we can't deduce $p(j)$

$\therefore$ Hence the given argument is not valid.

4) $\forall x\,[p(x) \to q(x)]$
$\forall x\,[q(x) \to r(x)]$
$\therefore \forall x\,[p(x) \to r(x)]$

$\Leftrightarrow$ $p(a) \to q(a)$, $q(a) \to r(a)$ $\Rightarrow p(a) \to r(a)$ (by syllogism)

$\therefore \forall x\,[p(x) \to r(x)]$

Steps	Reasons
1) $\forall x\,[p(x) \to q(x)]$	premise
2) $p(a) \to q(a)$	step ① & Rule of universal specialization
3) $\forall x\,[q(x) \to r(x)]$	premise
4) $q(a) \to r(a)$	step ③ & U.S.
5) $p(a) \to r(a)$	step ③ & ④ & syllogism
6) $\forall x\,[p(x) \to r(x)]$	step ⑤ & Rule of universal generalization

$\therefore$ The given argument is valid.

5. prove the foll. statement is valid.

$\forall x\,[p(x) \vee q(x)]$
$\exists x \sim p(x)$
$\forall x\,[\sim q(x) \vee r(x)]$
$\forall x\,[s(x) \to \sim r(x)]$
$\therefore \exists x \sim s(x)$

$\Leftrightarrow$ $p(a) \vee q(a)$, $\sim p(a)$ — $q(a)$; $\sim q(a) \vee r(a)$ — $\sim q(a) \vee r(a)$ } $r(a)$; $s(a) \to r(a)$ — $s(a) \to \sim r(a)$ } $s(a) \to \sim r(a)$

$\therefore \sim s(a)$

Steps	Reasons
1) $\forall x\; p(x) \vee q(x)$	premise
2) $\exists x \sim p(x)$	premise
3) $p(a) \vee q(a)$	① & U.S
4) $\sim p(a)$	③
5) $q(a)$	3, 4 & disjunctive syllogism
6) $\forall x \sim q(x) \vee r(x)$	premise
7) $\sim q(a) \vee r(a)$	(6) & U.S
8) $r(a)$	(5) & (7) & disjunctive syllogism
9) $\forall x\; s(x) \to \sim r(x)$	premise
10) $s(a) \to \sim r(a)$	⑨ & U.S.
11) $\sim s(a)$	8, 10 & Modus Tollens
12) $\therefore \exists x \sim s(x)$	(11) & U.G.

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Methods of proofs and Disproofs :-

Given a conditional $p \to q$ the process of establishing that the conditional is true by using the rules of logic and other known facts constitutes a proof of the conditional.

The process of establishing that a conditional is false is called as disproof.

Types of proofs :-

1. **Direct proofs :-** The direct proof of proving the conditional $p \to q$ is true is

a) Hypothesis :- First assume that p is true

b) Analysis :- Starting with hypothesis, employing the rule of logic and other known facts, infer that q is true.

c) conclusion :- $p \to q$ is true.

2. **Indirect proof :-** Steps are

a) $p \to q \Leftrightarrow \sim q \to \sim p$

b) Assume $\sim q$ is true

c) with the help of rule of logic and other known facts infer p is false. $\therefore \sim p$ is true.

d) If $\sim p$ and $\sim q$ is true, $\sim q \to \sim p$ is true
$\therefore p \to q$ is also true.

c) **proof by contradiction :-** Steps are

a) Hypothesis :- Assume that $p \to q$ is false
$p \to q$ is false only if p is true & q is false

b) Analysis :- Starting with the hypothesis that q is false, employing rules of logic and other known facts, infer that p is false. This contradicts that assumption that p is true.

c) Conclusion :- we infer that $p \to q$ is true because of the contradiction arrived in the analysis step.

Types of Disproofs :-

1. Disproof by contradiction :- We proved that the conditional $p \to q$ is false

(i) Hypothesis :- Assume that p is true & q is true & hence $p \to q$ is true

(ii) Analysis :- Using laws of logic and other known facts show that our assumption is wrong and hence $p \to q$ is false. This disproves the given statement.

2. Disproof by counter example :-

we know that the quantified statement $\forall x\, p(x)$ is false if for any one element 'a' is false. Hence take one case such that $p(a)$ is false and hence the given proposition $p(x)$ is false

is false.

Problems :-

1. Give direct proof of the statement " The square of an odd integer is an odd integer"

Soln: Assume that 'n' is an odd integer.

Then $n = 2k+1$ for some integer.

$\therefore n^2 = (2k+1)^2 = 4k^2+4k+1$

$= 2(2k^2+2k)+1$

Let $p = 2k^2+2k$ where p is some integer

$\therefore n^2 = 2p+1$

This proves that n^2 is an odd integer.

2. prove that for all integers 'k and l' if k & l are both odd then $(k+l)$ is even.

Soln: Take any two integers k & l and Assume that they are odd

$\therefore k = 2m+1$ & $l = 2n+1$ for some integers m & n

$\therefore k+l = (2m+1)+(2n+1)$

$= 2m+2n+2 = 2(m+n+1)$

$\therefore k+l$ is even number.

3. Given (i) a direct proof

(ii) An indirect proof

(iii) proof by contradiction for the following

"If m is an even integer then $(m+7)$ is odd.

Soln: (i) Direct proof.

Let p: m is even   q: m+7 is odd

Assume q is true   i.e. m+7 is odd

$\therefore m+7 = 2k+1 \Rightarrow m = 2k+1-7 \Rightarrow m = 2k-6$

$\Rightarrow m = 2(k-3)$

which is divisible by 2

Hence m is even $\therefore p \to q$ is true

$\because$ m+7 is odd
General form is 2k+
$\therefore m = 2(k-3)$ is even.

This proves the truth of the given statement.

(ii) Indirect proof :-

we need to prove that $\sim q \rightarrow \sim p$ is true

Assume $\sim q$ is true $\Rightarrow q$ is false $\Rightarrow m+7$ is even

$\therefore m+7 = 2k \Rightarrow m = 2k-7$

$\Rightarrow m$ is not divisible by 2 (or) m is odd

$\therefore p$ is false $\Rightarrow \sim p$ is true

$\therefore \sim q \rightarrow \sim p$ is true

Thus the given statement is proved using indirect proof

(iii) proof by contradiction :-

Assume $p \rightarrow q$ is false.

This is possible if p is true & q is false

Now if q is false, m+7 is not odd.

$\Rightarrow m+7$ is even

ie $m+7 = 2k \Rightarrow m = 2k-7$

$m = 2k-8+1$

$= 2(k-4)+1$

$\Rightarrow m$ is not divisible by 2 or m is odd

$\Rightarrow p$ is false.

$\therefore$ This contradicts the assumption that p is true

$\therefore p \rightarrow q$ can't be false

using proof by contradiction we proved that $p \rightarrow q$ is true.

4. Give (i) A direct proof (ii) Indirect proof (iii) Proof by contradiction for "If n is an odd integer, then n+9 is an even integer"

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Sub: Discrete Mathematical Structure.

Subcode: BCS405A

Module 2 :- Properties of the Integers.

- Mathematical Induction
- The well ordering principle - Mathematical Induction
- Recursive Definitions
- Fundemental principles of counting : The Rules of sum and product, permutations, Combinations, the binomial theorem, combinations with Repetition.

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Properties of Integers:-

The set Z represents the set of all integers (both +ve & -ve). A subset of this representing the set of all +ve integers denoted by Z^+ play a significant role in establishing certain results.

Given two distinct integers x & y satisfy either of two inequalities $x>y$ or $y>x$. ($x<y$ or $y<x$)

$\therefore Z^+ = \{ x / x \in Z, x$

$Z^+ = \{x / x \in Z, x>0\} = \{x / x \in Z, x \geq 1\}$.

Every non-empty subset of Z^+ say Z_0^+ contains an integer x_0 such that $x_0 \leq x \ \forall x \in Z_0^+$

i.e Z_0^+ contains a least / smallest element.

The well ordering principle:-

Every non-empty subset of the set of all +ve integers Z^+ contains a smallest element is said to be well-ordered.

The principle of Mathematical Induction:-

Statement:- Let S(n) be an open statement where $n \in Z^+$, satisfy the following conditions

(i) S(n) is true for n=1 or S(1) is true

(ii) If S(n) is true for n=k (say) $\in Z^+$ Then S(k+1) is true.

+ Equivalently, whenever $S(k)$ is true then $S(k+1)$ is also true. Then $S(n)$ is true for all $n \in Z^+$.

+ Working procedure:- Steps to establish that a given statement $S(n)$ is true for all integers $n \geq 1$.

Step 1:- verify that $S(1)$ is true

Step 2: we assume that $S(n)$ is true for an arbitrary integer $k \geq 1$.

This is equivalent to rewriting $S(n)$ replace $n = k$.

Starting from this step we show that $S(k+1)$ is true.

Generally, we achieve the result we add $(k+1)^{th}$ term on both sides of $S(k)$ and simplify the RHS to show that $S(k+1)$ is true.

Finally, we conclude that $S(n)$ is true for all $n \geq 1$ by the principle of Mathematical Induction.

Problems:-

I. prove the following statements by mathematical induction.

1. $1+r+r^2+ \cdots +r^{n-1} = \dfrac{1-r^n}{1-r}$; $n \geq 1$, $r \neq 1$, $n \in Z^+$

Sol: Let $S(n): 1+r+r^2+ \cdots +r^{n-1} = \dfrac{1-r^n}{1-r}$

$S(n) = \dfrac{1-r^n}{1-r}$

$S(1) = \dfrac{1-r}{1-r} = 1$

Step 1: $S(1)$: LHS $= 1$ & RHS $= \dfrac{1-r}{1-r} = 1$

$\therefore S(1)$ is true

Step 2: We shall assume that $S(n)$ is true for $n=k$.

$S(k): 1+r+r^2+ \cdots +r^{k-1} = \dfrac{1-r^k}{1-r}$ —— ①

We shall add the term r^k on to b.s. of ①

i.e $1+r+r^2+ \cdots +r^{k-1}+r^k = \dfrac{1-r^k}{1-r} + r^k$

$\Rightarrow 1+r+r^2+ \cdots +r^k = \dfrac{1-r^k+r^k(1-r)}{1-r}$

$= \dfrac{1-r^k+r^k-r^{k+1}}{1-r}$

$\Rightarrow 1+r+r^2+ \cdots +r^{(k+1)-1} = \dfrac{1-r^{k+1}}{1-r}$ —— ②.

Comparing ① & ②, we conclude that $S(k+1)$ is true.

Thus, by the principle of Mathematical Induction, $S(n)$ is true for all $n \geq 1$, $r \neq 1$.

2. $\sum_{i=1}^{n} i = 1+2+3+ \cdots\cdots + n = \frac{n(n+1)}{2}$ [Jan 2010]

Sol? Let $S(n): 1+2+3+ \cdots\cdots + n = \frac{n(n+1)}{2}$

Step 1: $S(1)$: LHS $= 1$ & RHS $= \frac{1(1+1)}{2} = \frac{2}{2} = 1$ $S(n) = \frac{n(n+1)}{2}$

$\therefore$ $S(1)$ is true. $S(1) = \frac{2}{2} = 1$

Step 2: We shall assume that $S(n)$ is true for $n=k$

$S(k): 1+2+3+ \cdots + k = \frac{k(k+1)}{2}$ —— ①

We shall add the term $(k+1)$ on b.s. of ①

i.e $1+2+3+ \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$

$\Rightarrow 1+2+3+ \cdots + k + (k+1) = \frac{k+1}{2}[k+2]$

$= \frac{(k+1)}{2}[(k+1)+1]$ —— ②

Comparing ① & ②, we conclude that $S(k+1)$ is true.

Thus by the principle of mathematical induction $S(n)$ is true for all $n \geq 1$.

3. $1.3 + 2.4 + 3.5 + \cdots\cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Sol? Let $S(n)$: $1.3 + 2.4 + 3.5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Step 1: $S(1)$: LHS $= 1.3 = 3$ & RHS $= \frac{1(2)(9)}{6} = 3$

$\therefore$ $S(1)$ is true.

Step 2: We shall assume that $S(n)$ is true for $n=k$.

$1.3 + 2.4 + 3.5 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}$ —— ①

We shall add the term $(k+1)(k+3)$ onto b.s. of ①. (or) ~~Replace k by k+1 on b.s.~~

$$1.3 + 2.4 + \cdots + k(k+2) + (k+1)(k+3)$$

$$= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$

$$= \frac{(k+1)}{6}\left[2k^2 + 7k + 6k + 18\right]$$

$$= \frac{(k+1)}{6}\left[2k^2 + 13k + 18\right]$$

$$= \frac{(k+1)}{6}(k+2)(2k+9) \text{, by factorization}$$

$$= \frac{(k+1)(\overline{k+1}+1)(2\overline{k+1}+7)}{6} \quad \text{——— ②}$$

Comparing ① & ②, we conclude that $S(k+1)$ is true.

4. $1.2.3 + 2.3.4 + 3.4.5 + \cdots + n(n+1)(n+2)$
$= \frac{1}{4} n(n+1)(n+2)(n+3)$

Sol^n. Let $S(n)$: $1.2.3 + 2.3.4 + 3.4.5 + \cdots + n(n+1)(n+2)$
$= \frac{1}{4} n(n+1)(n+2)(n+3)$

Step 1: $S(1)$: LHS $= 1.2.3 = 6$

RHS $= \frac{1}{4} 1(2)(3)(4) = 6$

$\therefore S(1)$ is true.

Step 2: We shall assume that $S(n)$ is true for $n = k$.

$$1.2.3 + 2.3.4 + \cdots + k(k+1)(k+2) = \frac{1}{4} k(k+1)(k+2)(k+3) \quad \text{——— ①}.$$

We shall add the term $(k+1)(k+2)(k+3)$ onto b.s. of ①.

$$1.2.3 + 2.3.4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$$

$$= \frac{1}{4} k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$\Rightarrow 1.2.3 + 2.3.4 + \cdots + (k+1)(\overline{k+1}+1)(\overline{k+1}+2)$$

$$= \frac{(k+1)(\overline{k+1}+1)(\overline{k+1}+2)(\overline{k+1}+3)}{4} \quad \text{—— ②}$$

Comparing ① & ②, we conclude that $S(k+1)$ is true.

Thus, by the principle of mathematical induction, $S(n)$ is true for all $n \geq 1$

Imp 5. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ [[illegible]e 2010]

Soln: we have to prove that,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Let $S(n)$: $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1: $S(1)$: LHS $= 1^2 = 1$ & RHS $= \frac{(1)(2)(3)}{6} = 1$

$\therefore S(1)$ is true.

Step 2: we shall assume that $S(n)$ is true for $n = k$

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{—— ①}$$

We shall add the term $(k+1)^2$ on b.s. of ①

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)}{6}\left[2k^2+k+6(k+1)\right]$$

$$= \frac{(k+1)}{6}\left[2k^2+7k+6\right]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$1^2+2^2+3^2+ \cdots +(k+1)^2 = \frac{(k+1)(\overline{k+1}+1)(2\overline{k+1}+1)}{6} \quad \text{—— (2)}$$

Comparing (1) & (2), we conclude that $S(k+1)$ is true.

Imp 6. $\displaystyle\sum_{i=1}^{n} i(2^i) = 2+(n-1)2^{n+1}$ [Dec. 2011]

Sol: Let $S(n): 1(2^1)+2(2^2)+3(2^3)+ \cdots +n(2^n)$
$= 2+(n-1)2^{n+1}$

Step 1: LHS $= 1(2^1) = 2$ &

RHS $= 2+0 = 2$

$\therefore$ $S(1)$ is true.

Step 2: We shall assume that $S(n)$ is true for $n=k$.

$1(2^1)+2(2^2)+3(2^3)+ \cdots +k(2^k) = 2+(k-1)2^{k+1}$ —— (1)

We shall add the term $(k+1)2^{k+1}$ on to b.s. of (1)

$1(2^1)+2(2^2)+ \cdots +k(2^k)+(k+1)2^{k+1} = 2+(k-1)2^{k+1} + (k+1)2^{k+1}$

$$= 2+2^{k+1}\left[k-1+k+1\right]$$

$$= 2+2^{k+1}\cdot 2k$$

$$= 2+2^{k+2}\cdot k$$

$$= 2+(\overline{k+1}-1)\,2^{(k+1)+1} \quad \text{—— (2)}$$

Comparing (1) & (2), we conclude that $S(k+1)$ is true.

Thus, by the principle of MI, $S(n)$ is true for all $n \geq 1$.

v.Imp. $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}, \forall n \in Z^+$ [June 2012]

Sol? Let $S(n): \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step-1: $S(1)$: $LHS = \frac{1}{1.2} = \frac{1}{2}$ $RHS = \frac{1}{1+1} = \frac{1}{2}$

$\therefore S(1)$ is true

Step-2: we shall assume that $S(n)$ is true for $n=k$

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1} \quad \text{——} (1)$$

we shall add the term $\frac{1}{(k+1)(k+2)}$ onto both sides of (1)

ie $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{1}{(k+1)}\left[k + \frac{1}{k+2}\right]$$

$$= \frac{1}{(k+1)}\left[\frac{k^2+2k+1}{k+2}\right] = \frac{1}{\cancel{k+1}}\left[\frac{(k+1)^{\cancel{2}}}{k+2}\right]$$

$$= \frac{k+1}{k+2}$$

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{(k+1)(\overline{k+1}+1)} = \frac{k+1}{(k+1)+1} \quad \text{——} (2)$$

comparing (1) & (2), we conclude that $S(k+1)$ is true.

8. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Sol? Let $S(n)$: $1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$

Step 1: For $S(1)$ $LHS = 1^3 = 1$

$RHS = 1^2 \frac{(1+1)^2}{4} = 1$

$\therefore S(1)$ is true

Assume $S(n)$ is true for $n=k$

$$1^3+2^3+3^3+\dots\dots+k^3 = \frac{k^2(k+1)^2}{4} \quad \text{--- (1)}$$

Add $(k+1)^3$ on b.s.

$$1^3+2^3+3^3+\dots\dots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$= \frac{(k+1)^2}{4}[k^2+4(k+1)]$$

$$= \frac{(k+1)^2}{4}[k^2+4k+4]$$

$$= \frac{(k+1)^2}{4}(k+2)^2$$

$$= \frac{(k+1)^2}{4}[(k+1)+1]^2 \quad \text{--- (2)}$$

Comparing (1) & (2),

We conclude that $S(k+1)$ is true.

Thus by M.I, $S(n)$ is true for all $n \geq 1$.

V Imp 9. 5 divides (n^5-n) for every positive integer. (July 2013)

Sol^n. Let $S(n)$: (n^5-n) is divisible by 5.

Step 1: $S(1)$: $(1^5-1)=0$ is divisible by 5

$\therefore$ $S(1)$ is true.

Step 2: We shall assume that $S(n)$ is true for $n=k$.

That is (k^5-k) is divisible by 5 --- (1).

$S(k)$: $(k^5-k) = 5a$, $a \in Z$

consider, $S(k+1)$: $(k+1)^5-(k+1)$

$S(k+1)$: $(k^5+5k^4+10k^3+10k^2+5k+1)-(k+1)$

: $(k^5-k)+5(k^4+2k^3+2k^2+k)$

: $5a+5(k^4+2k^3+2k^2+k)$

: $5(a+k^4+2k^3+2k^2+k)$

$S(k+1)$: $5b$ (say) , $b \in Z$ ⟶ (3)

(1) & (2) $S(k+1)$ is true

$\therefore$ $S(n)$ is true for all +ve integers.

V.Imp 10. Show that if $n \geq 24$, then n can be written as sum of 5's and or 7's for all $n \in Z^+$

(Jan 2017)

Sol^n. Let $S(n)$: $n \geq 24$ & n can be written as sum of 5's and or 7's, $n \in Z^+$

Step 1: Since $n \geq 24$, we shall consider $n = 24$ initially

$$24 = (5+5) + (7+7)$$
$$24 = 2(5) + 2(7)$$

$\therefore$ $S(24)$ is true.

Step 2: We shall assume that $S(n)$ is true for $n = k$.

That is $k = p(5) + q(7)$ where $p, q \in Z$ & $p>2, q>2$ —— ①

Now, $k+1 = [p(5) + q(7)] + 1$

$$k+1 = [p(5) + (q-2+2)7] + 1$$

$$k+1 = p(5) + (q-2)7 + 14 + 1$$
$$= p(5) + (q-2)7 + 15$$
$$= p(5) + (q-2)7 + 3(5)$$
$$k+1 = (p+3)5 + (q-2)7 \quad , p+3, q-2 \in Z \quad \text{—— ②}$$

Comparing ① & ②, we conclude that $S(k+1)$ is true

$\therefore$ By M I $S(n)$ is true for all $n \in Z^+$

V.Imp 11. Let $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. Prove that $a_n \leq 3^n$ for all positive integers n.

(June 2014)

Sol^n. Let $S(n)$: $a_n \leq 3^n$

Step 1: $a_0 = 1 \leq 3^0$, $a_1 = 2 < 3$, $a_2 = 3 < 3^2 = 9$

$S(0)$, $S(1)$, $S(2)$ are all true.

Step 2: We shall assume that $S(n)$ is true for $n = k$

where $n = 0, 1, 2, 3, \ldots k$ $k \geq 2$

That is, $S_k : a_k \leq 3^k$ —— (1)

Now $a_{k+1} = a_k + a_{k-1} + a_{k-2}$ ($\because$ by def^n of a_n)

W.K.T. $a_k \leq 3^k$ $3^{k-1} \leq 3^k$, $3^{k-2} \leq 3^k$ is also true

$$a_{k+1} \leq 3^k + 3^k + 3^k$$

$$\leq 3 . 3^k$$

$$a_{k+1} \leq 3^{k+1} \quad \text{——} \quad (2)$$

(1) & (2), $S(k+1)$ is true.

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Recursive Definitions :-

An ordered set of real numbers $a_1, a_2, a_3 \cdots a_n \cdots$ is called a sequence and it is denoted by $\{a_n\}$. a_n is called the n^{th} term or the general term of the sequence.

The Arithmatic progression
$a, a+d, a+2d, \cdots$ is $a+(n-1)d$

Ex:

i) $1, 3, 5, \cdots$ represents the seq $\{2n-1\}$

ii) $1, -1, 1, -1, \cdots$ — " — $\{(-1)^{n-1}\}$

iii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ — " — $\{\frac{1}{n}\}$

A sequence represented by two methods

i) Explicit method

ii) Recursive method.

In the explicit method the general term of the sequence is explicitly given. The various values of the sequence are obtained by given values for $n \in Z^+$

In the Recursive method the general term relates to a few of its predecessors. Equivalently first few terms of the sequence are explicitly given and the general term being generated according to a rule. This well facilitate to obtain new terms of the sequence from the already known terms.

Examples :-

1. $1, 3, 5, 7, \ldots\ldots$

Here the n^{th} or general term is $a+(n-1)d$

$$= 1+(n-1)2$$
$$= 1+2n-2 = 2n-1$$

By explicit method the sequence is $\{2n-1\}$

By Recursive method

$a_1 = 1,\ a_2 = 3,\ a_3 = 5,\ a_4 = 7, \ldots\ldots$

$$a_2 = 1+2 = a_1 + 2$$
$$a_3 = 3+2 = a_2 + 2$$
$$a_4 = 5+2 = a_3 + 2$$
$$\ldots\ldots$$
$$a_n = a_{n-1} + 2,\ n \in Z^+,\ n > 1$$

2. $35, 30, 25, 20, \ldots\ldots$

Here the n^{th} or general term is $a+(n-1)d$

$$= 35 + (n-1)(-5)$$
$$= 35 - 5n + 5$$
$$= -5n + 40$$
$$= 40 - 5n$$

As per explicit method the seq is $\{40-5n\}$.

Next, by recursive

$a_1 = 35,\ a_2 = 30,\ a_3 = 25,\ a_4 = 20, \ldots\ldots$

$$a_2 = 35-5 = a_1 - 5$$
$$a_3 = 30-5 = a_2 - 5$$
$$a_4 = 25-5 = a_3 - 5$$
$$\ldots\ldots$$
$$a_n = a_{n-1} - 5,\ n \in Z^+,\ n > 1$$

Problems:-

1. If $\{a_n\}$ represents the sequence of integers having $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and satisfying $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \in Z^+$, $n \geq 3$. compute the value of a_6.

Sol^n: By data $a_0 = 1$, $a_1 = 2$, $a_2 = 3$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{—} ①$$

put $n = 3, 4, 5, 6$ to ①

$$a_3 = a_2 + a_1 + a_0 = 3 + 2 + 1 = 6$$

$$a_4 = a_3 + a_2 + a_1 = 6 + 3 + 2 = 11$$

$$a_5 = a_4 + a_3 + a_2 = 11 + 6 + 3 = 20$$

$$a_6 = a_5 + a_4 + a_3 = 20 + 11 + 6 = 37.$$

Thus $\boxed{a_6 = 37}$

V Imp

2. obtain the recursive definition for the sequence $\{a_n\}$ in each of the following (DEC. 2012)

(i) $a_n = 5n$ (ii) $a_n = 2 - (-1)^n$

Sol^n: (i) $a_n = 5n$

$a_1 = 5$, $a_2 = 10$, $a_3 = 15$, $a_4 = 20$, - - - - .

$$a_2 = 5 = 5 + 5 = a_1 + 5$$

$$a_3 = 15 = 10 + 5 = a_2 + 5$$

$$a_4 = 20 = 15 + 5 = a_3 + 5$$

. . . .

$a_n = a_{n-1} + 5$ where $a_1 = 5$, & $n > 1$, $n \in Z^+$

(ii) $a_n = 2 - (-1)^n$ —— ①

$a_1 = 3$, $a_2 = 1$, $a_3 = 3$, - - - . $a_n = 2 - (-1)^n$

and $a_{n+1} = 2 - (-1)^{n+1}$ —— ②

② − ① ⇒

$$a_{n+1} - a_n = (2 - (-1)^{n+1}) - (2 - (-1)^n)$$

$$= \cancel{2} - (-1)^{n+1} - \cancel{2} + (-1)^n$$

$$= (-1)^n [-(-1) + 1] = (-1)^n [(-1)^2 + 1] = 2(-1)^n$$

Thus $a_1 = 3$ and $a_{n+1} = a_n + 2(-1)^n$ for $n \geq 1$ is a recursive defn of the given sequence.

(iii) $a_n = 6^n$

Here $a_1 = 6$, $a_2 = 6^2$, $a_3 = 6^3$, $a_4 = 6^4$, ...

we can write

$a_1 = 6$

$a_2 = 6.6 = 6a_1$

$a_3 = 6.6^2 = 6a_2$ ---- $a_n = 6a_{n-1}$

& $a_{n+1} = 6a_n$. for $n \geq 1$

is the recursive defn of the given seq

(iii) $a_n = 3n+7$

$a_1 = 10$, $a_2 = 13$, $a_3 = 16$, $a_4 = 19$, ...

we can write $a_1 = 10$

$a_2 = 13 = 10+3 = a_1 + 3$

$a_3 = 16 = 13+3 = a_2+3$

$a_4 = 19 = 16+3 = a_3 + 3$

$a_n = a_{n-1} + 3$ for $n \geq 2$

(iv) $a_n = n(n+2)$

$a_1 = 3$, $a_2 = 8$, $a_3 = 15$, $a_4 = 24$, ...

$a_2 - a_1 = 5 = 2\times1+3$

$a_3 - a_2 = 7 = 2\times2+3$

$a_4 - a_3 = 9 = 2\times3+3$

....

$a_{n+1} - a_n = 2\times n+3 = 2n+3$

$a_{n+1} = (2n+3) + a_n$

Thus $a_1 = 3$ & $a_{n+1} = a_n + (2n+3)$ is the recursive defn of a given seq.

(V) $a_n = n^2$

$a_1 = 1^2,\ a_2 = 2^2,\ a_3 = 3^2,\ a_4 = 4^2, \ldots$

$a_2 - a_1 = 3 = 2\times1+1$

$a_3 - a_2 = 5 = 2\times2+1$

$a_4 - a_3 = 7 = 2\times3+1$

$\ldots\ldots$

$a_{n+1} - a_n = 2n+1$

$a_{n+1} = a_n + (2n+1)$ for $n \geq 1$ is the req. seq.

thus $a_1 = 1$,

VImp 3. Find an explicit formula for $a_n = a_{n-1} + n,\ a_1 = 4$ for $n \geq 2$. (Dec. 2013, 2014)

Soln: By data: $a_n = a_{n-1} + n$ —— ①

Put $n = n-1, n-2, n-3 \ldots$

$a_{n-1} = a_{n-2} + (n-1)$

$a_{n-2} = a_{n-3} + (n-2)$

$a_{n-3} = a_{n-4} + (n-3)$

① $\Rightarrow$ $a_n = [a_{n-2} + (n-1)] + n$

$= [a_{n-3} + (n-2)] + (n-1) + n$

$= [a_{n-4} + (n-3)] + (n-2) + (n-1) + n$

$= a_{n-4} + (n-3) + (n-2) + (n-1) + n$ $\quad n>4$ i.e $n=5$

$\ldots\ldots$

$\ldots\ldots$

$= a_1 + 2 + 3 + 4 + \cdots + n$

$a_n = (a_1 - 1) + (1 + 2 + 3 + 4 + \cdots + n)$

Using $a_1 = 4$, we get

$a_n = 3 + \frac{1}{2}n(n+1)$ $\quad$ ($\because$ by std. result)

is the req. explicit formula for a_n.

$2+2^2+2^3+\cdots$

V.IMP 4. Find an explicit defn of the sequence defined recursively by $a_1=7$, $a_n = 2a_{n-1}+1$ for $n \geq 2$. (Jan 17)

Soln: $a_n = 2a_{n-1}+1$ —— ①

put $n = n-1, n-2, n-3, \ldots$

$a_{n-1} = 2a_{n-2}+1$

$a_{n-2} = 2a_{n-3}+1$

$a_{n-3} = 2a_{n-4}+1$

①$\Rightarrow$ $a_n = 2(2a_{n-2}+1)+1 = 2.2a_{n-2}+2+1$

$= 2.2(2a_{n-3}+1)+2+1 = 2.2.2a_{n-3}+2^2+2+1$

$= 2.2.2(2a_{n-4}+1)+2^2+2+1$

$a_n = 2^4 a_{n-4}+2^3+2^2+2+1$

$\cdots\cdots$

$2^{5-1}a_{n-(5-1)}$ $\cdots + 2^n + 2^{n-1}+2^{n-2}+1$

$2^{n-1}a_{n-(n-1)}$

$a_n = 2^{n-1}a_1 + 1+2+2^2+\cdots+2^{n-3}+2^{n-2}$

$= 2^{n-1}a_1 + \dfrac{2^{n-1}-1}{2-1}$

Using $a_1 = 7$

& $1+a+a^2+\cdots+a^{n-1} = \dfrac{a^n-1}{a-1}$ for $a>1$

this becomes $a_n = 7\times 2^{n-1}+(2^{n-1}-1)$

$= 7\times 2^{n-1}+(2^{n-1}-1)$

$= 8\times 2^{n-1}-1$, $n\geq 2$

$= 2^{n-1}(7+1)-1$

5. Find a unique soln of the recurrence relation: $4a_n - 7a_{n-1} = 0$, $n \geq 1$ & $a_0 = 1$

Soln: By data: $4a_n - 7a_{n-1} = 0$

$\Rightarrow a_n = \frac{7}{4}a_{n-1}$

$= \frac{7}{4}\left[\frac{7}{4}a_{n-2}\right] = \left(\frac{7}{4}\right)^2\left[\frac{7}{4}a_{n-3}\right]$

$a_n = \frac{7}{4} a_{n-1}$

$a_{n-1} = \frac{7}{4} a_{n-2}$

$a_{n-2} = \frac{7}{4} a_{n-3}$

$a_{n-3} = \frac{7}{4} a_{n-4}$

$$= \left(\frac{7}{4}\right)^3 \left[\frac{7}{4} a_{n-4}\right]$$

$$a_n = \left(\frac{7}{4}\right)^4 a_{n-4}$$

$$a_n = \left(\frac{7}{4}\right)^n a_0 \quad \text{since } a_0 = 1$$

$$\therefore \; a_n = \left(\frac{7}{4}\right)^n, \; a_0 = 1 \quad n \geq 1$$

6. Given that r is fixed constant derive an explicit formula for

(i) $U_n = U_{n-1} - r$, $u_1 = 10$, $n \in Z^+$ & $n > 1$

Sol^n: $U_n = (U_{n-1} - r)$ by back subst^n

$= (U_{n-2} - r) - r$

$= (U_{n-2} - 2r)$

$= (U_{n-3} - r) - 2r = U_{n-3} - 3r$

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etc.

$U_n = U_{n-4} - 4r$

~~$U_n = 10 - ($~~

$U_n = U_1 - (n-1)r$

$= 10 - (n-1)r \quad n > 1$

$U_n = (10 + r) - nr$

(ii) $U_n = U_{n-1} + r^n$ $\quad U_1 = 5, \; n > 1$ by back subst^n

$U_n = (U_{n-2} + r^{n-1}) + r^n$

$= U_{n-2} + (r^{n-1} + r^n)$

$U_n = U_{n-3} + r^{n-2} + r^{n-1} + r^n \quad n=4$

Now $U_n = U_1 + (r^3 + r^4 + r^2 \ldots + r^{n-1} + r^n)$

$= U_1 + (r^2 + r^3 + r^4 + \cdots + r^{n-1} + r^n)$

$= 5 + r^2(1 + r + r^2 + \cdots + r^{n-2})$

$= 5 + r^2\left[\frac{r^{n-1} - 1}{r - 1}\right], \; r > 1, \; n > 1$ is req. formula

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Fundamental principles of counting:-

The word count is nothing but finding the total number. It is an act of counting. Techniques of counting is essential both in mathematics/DMS and computer science.

We first discuss two basic rules of counting

1. SUM Rule

Let A_1 and A_2 be two events, where the event A_1 can happen in m_1 ways and event A_2 can happen in m_2 ways. Further the events A_1, A_2 can't happen at the same time. Then either of the two events (A_1 or A_2) can happen in (m_1+m_2) ways.

In general, if the events $A_1, A_2, A_3 - - - A_n$ can happen in $m_1, m_2, m_3 \ldots m_n$ ways. such that no two events happen simultaneously, then the number of ways of the happening of one of the events (A_1, or A_2 or A_3 - - -. or A_n) is $(m_1+m_2+m_3+ - - - +m_n)$.

A college has 7 female professor & 6 male professors. Student can choose a professor in $7+6=13$ ways.

2. Product Rule

Let A_1 and A_2 be two events which happens one after another. If the event A_1 happens in m_1 ways and for each of these A_2 happens in m_2 ways then both the events happens in $m_1 . m_2$ ways.

Permutations and Combinations:-

Permutation :- means arrangement of things where the word arrangement is to be understood in the perspective that the order of things is to be necessarily considered.

Permutation is an arrangement of things in a definite order of number of objects taken some or all at a time. $p(n,r) = \frac{n!}{(n-r)!}$

Combination is a selection that can be made by taking some or all of a no of things is called comb?

Combination means selection of things where it is evident that the order is not in consideration.

Permutation of n things taken r at a time is to be understood as the arrangement of r things in all the possible ways. where $r \leq n$.

This is denoted by nP_r or $P(n, r)$

Combination of n things taken r at a time is to be understood as the selection of r things out of n things where $r \leq n$. This is denoted by nC_r or $\binom{n}{r}$ or $C(n, r)$

Things	permutation	combination
a, b	a, b & ba = 2 ways i.e. $2P_2 = 2$	ab or ba they are same = 1 $2C_2 = 1$
a, b, c	ab, ba, bc, cb, ac, ca = 6 ways $3P_2 = 6$	ab or bc or ca = 3 ways $3C_2 = 3$
5, 6, 7, 8	$4P_3 = 24$	$4C_3 = 4$ ways

Note:

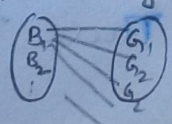
② $nP_r = \frac{n!}{(n-r)!}$ (without repetition)

① $p(n,r) = nP_r = n!$ with repetions
$= n(n-1)(n-2) \cdots (n-(r-1))$

③ $nC_r = \frac{n!}{r!\,(n-r)!}$ ④ $\Rightarrow nC_r = \frac{nP_r}{r!}$

combn ex: suppose we have diff objects A, B, C & D. Selecting 2 objects at a time AB, AC, AD, BC, BD, CD

A tennis club consists of 8 boys & 5 girls. In how many ways (A_1, A_2) can a mixed double team be choosen?



$nP_r = n!$ $\quad nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-r)n!}{r!(n-r)!} = \frac{nP_r}{r!}$

$nC_r = \frac{nP_r}{r!} = \frac{n!}{r!}$

Note:

1) $2P_2 = 2(2-1) = 2$ $\qquad 2C_2 = \frac{2(2-1)}{2!} = 1$

2) $3P_3 = 3(3-1)(3-2)$
$= 3.2.1 = 6$ $\qquad 3C_3 = \frac{3(3-1)}{3!} = \frac{6}{6} = 1$

$3P_2 = 3(2) = 6$ $\qquad 3C_2 = \frac{3.2}{2!} = 3$

3) $4P_3$
$= 4(4-1)(4-2)$ $\qquad 4C_3 = \frac{4.3.2}{2!} = 4$
$= 24$

& $8P_5 = 8.7.6.5.4 = 6720$

$6P_4 = 6.5.4.3 = 360$

4) $10C_4 = \frac{10.9.8.7}{4!} = 210$

$15C_3 = \frac{15.14.13}{3!} = 455$

$15C_{12} = 15C_3$ by using $nC_r = nC_{n-r}$
$= 455.$

Problems:-

1. Find the value of n in the following cases:
(i) $nP_2 = 132$ $\quad$ (ii) $nP_3 = 3.\,nP_2$
(iii) $2P(n,2) + 50 = P(2n,2)$

Soln: (i) consider $nP_2 = 132$
$\Rightarrow n(n-1) = 132$
$\Rightarrow n^2 - n = 132$
$\Rightarrow n^2 - n - 132 = 0.$
$\Rightarrow n^2 - 12n + 11n - 132 = 0.$
$\Rightarrow n(n-12) + 11(n-12) = 0.$
$\Rightarrow (n-12).(n+11) = 0.$
$\therefore n = 12, -11$

Thus the required $n = 12.$

perm ex: No. of 3 letter words, with or without meaning, which can be formed out of the letter of the word 'NUMBER', where repetition of the letter is not allowed.

NUM, UMB, MBE, BER, - - -

$6 \times 5 \times 4 = 120$

(ii) $nP_3 = 3 \cdot nP_2$

$$\cancel{n}\cancel{(n-1)}(n-2) = 3\cancel{n}\cancel{(n-1)}$$

$$n-2 = 3$$

$$n = 5$$

Thus the required $n=5$

(iii) $2P(n,2) + 50 = P(2n,2)$ $\quad \div 2$

$$\cancel{2}n(n-1) + \cancel{50} = 2n(2n-1)$$

$n(2n-1) = 2n^2 - n$

$$n^2 - n + 25 = \cancel{2n^2 - 2n}$$

$$2n^2 - \cancel{2n} - n^2 + \cancel{n} - 25 = 0.$$

$$n^2 \cancel{-n} = 25$$

$$n = \pm 5$$

Thus the required $n=5$

(iv) P.T. $(n+1)P_r = (n+1) \cdot nP_{r-1}$

$nP_r = \frac{n!}{(n-r)!}$

Sol^n. $LHS = (n+1)P_r = \frac{(n+1)!}{[(n+1)-r]!}$

$$= (n+1)\frac{n!}{(n-r+1)!}$$

$$= (n+1) \cdot nP_{r-1} = RHS$$

(v) P.T. $nP_r = (n-r+1)\, nP_{r-1}$

Sol^n. $RHS = (n-r+1)\, nP_{r-1}$

$$= (n-r+1)\frac{n!}{[n-(r-1)]!}$$

$$= \cancel{(n-r+1)}\frac{n!}{\cancel{(n-r+1)}(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

$$= nP_r = LHS.$$

6. Prove that ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

<u>Sol:</u> $LHS = \dfrac{n!}{r!\,(n-r)!} + \dfrac{n!}{(r-1)!\,(n-(r-1))!}$

$$= \frac{n!}{r\,(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n-\cancel{r}+1+\cancel{r}}{r\,(n-r+1)}\right]$$

$$= \frac{n!\,(n+1)}{r\,(r-1)!\,(n-r)!\,(n-r+1)}$$

$$= \frac{n!\,(n+1)}{r!\,(n-r+1)!}$$

$$= \frac{(n+1)!}{r!\,(n+1-r)!} = {}^{n+1}C_r = RHS$$

7. Prove that ${}^{n}C_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$

<u>Sol:</u> $RHS = {}^{(n-1)}C_r + {}^{(n-1)}C_{r-1}$

$$= \frac{(n-1)!}{r!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-1-r+1)!}$$

$$= \frac{(n-1)!}{r!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-r)!}$$

$$= \frac{(n-1)!}{r\,(r-1)!\,(n-r-1)!} + \frac{(n-1)!}{(r-1)!\,(n-r)\,(n-r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left[\frac{1}{r} + \frac{1}{n-r}\right]$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left[\frac{n-\cancel{r}+\cancel{r}}{r\,(n-r)}\right]$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!} \quad \frac{n}{r(n-r)} \quad = \frac{n!}{r!\,(n-r)!}$$

$$= nC_r = LHS.$$

8. 6 boys and 4 girls got elected as class representatives in a college. A five members student council has to be formed from the elected CR's. In how many ways this council can be formed such that i) There are 3 boys and 2 girls
ii) Atleast 2 girls
iii) Atleast 3 boys

Sol^n: C.R's 6 boys (6B) & 4 Girls (4G)

i) 3B + 2G is possible in $6C_3 \times 4C_2$ ways

$$\frac{6.5.4}{3.2.1} \times \frac{4.3}{2.1}$$

$$\Rightarrow 20 \times 6 = 120 \text{ ways}$$

ii) Atleast 2 girls: The various options are

3B+2G, 2B+3G, 1B+4G

$\Rightarrow 6C_3 \times 4C_2$, $6C_2 \times 4C_3$, $6C_1 \times 4C_4$

$\Rightarrow \frac{6.5.4}{3.2.1} \times \frac{4.3}{2.1}$, $\frac{6!}{2!\,4!} \times \frac{4!}{3!\,1!}$, $\frac{6!}{1!\,5!} \times \frac{4!}{4!\,0!}$

$\Rightarrow 20 \times 6$, $\frac{{}^{3}\cancel{6}.5.\cancel{4.3.2.1}}{\cancel{2}.1.\,\cancel{4.3.2.1}} \times \frac{4.\cancel{3.2.1}}{\cancel{3.2.1}}$, $\frac{6.\cancel{5.4.3.2.1}}{1.\cancel{5.4.3.2.1}} \times 1$

$\Rightarrow 120$, 15×4, 6

$\Rightarrow 120, 60, 6$

The required number of ways is $120 + 60 + 6 = 186$

iii) $(3B+2G), (4B+1G), (5B+0G)$

$\Rightarrow 6C_3 \times 4C_2, \quad 6C_4 \times 4C_1, \quad 6C_5 \times 4C_0$

$\Rightarrow \frac{6!}{3!\,3!} \times \frac{4!}{2!\,2!}, \quad \frac{6!}{4!\,2!} \times 4, \quad \frac{6!}{5!\,1!} \times 1$

$\Rightarrow \frac{6.5.4.3.2.1}{3.2.1\;3.2.1} \times \frac{4.3.2.1}{2.1\;2.1}, \quad \frac{6.5.4.3.2.1}{4.3.2.1\;2.1} \times 4, \quad \frac{6.5.4.3.2.1}{5.4.3.2.1} \times 1$

$\Rightarrow 20 \times 6, \quad 15 \times 4, \quad 6 \times 1$

$\Rightarrow 120, 60, 6$

The required no. of ways is $120+60+6 = 186$

V.Imp 9. A certain question paper contains two parts A and B, each containing 4 questions. How many different ways a student can answer 5 questions by selecting atleast 2 from each part? (Dec 10)

Soln: part – A :- 4 questions. part B : 4 questions.

Atleast 2Q to be selected from each part to fulfill the requirement of 5Q. The options are (part A – 2Q & part B – 3Q) or (part A – 3Q & part B – 2Q) ie $4C_2 \times 4C_3$ or $4C_3 \times 4C_2$ each being 24 ways.

The required number of ways is equal to

$24+24 = 48$ ways.

10. If $nP_r = 5040$, $nC_r = 210$ find n & r

Soln: W.K.T. $nC_r = \frac{n!}{r!\,(n-r)!} = \frac{nP_r}{r!}$

$\Rightarrow r! = \frac{nP_r}{nC_r} = \frac{5040}{210} = 24$

$r! = 4! \Rightarrow \boxed{r=4}$

Consider $nC_r = 210$ since $r=4$

$\Rightarrow nC_4 = 210$

we have $n \geq 4$ & by taking $n = 4, 5, 6, \ldots$

$5C_4 = 5$, $6C_4 = 15$, $7C_4 = 35$, $8C_4 = 70$

$9C_4 = 126$, $10C_4 = 210$ Hence $n=10$

Thus $n=10$ and $r=4$.

11. In how many ways can three or more persons can be selected from twelve persons?

Sol^n: Here $n=12$ and $r = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

The number of ways are equal to

$12C_3 + 12C_4 + \ldots\ldots + 12C_{12}$

we note that $12C_{12} = 1$, $12C_{11} = 12C_1 = 12$

$12C_9 = 12C_3 = 220$

$12C_{10} = 12C_2 = 66$, $12C_7 = 12C_5 = 792$,

$12C_8 = 12C_4 = 495$

$12C_6 = 924$

$\Rightarrow (220 \times 2) + (495 \times 2) + (792 \times 2) + 924 + (66 + 12 + 1)$

$= 4095$

Thus the required number of ways is 4095

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Permutation with alike things

The number of permutation of n things taken all at a time as already discussed is $nP_n = n!$ Suppose that out of n things, n_1 things is of one type, n_2 things is of 2nd type, ... n_r things of r^{th} type. Where $n_1+n_2+\dots+n_r = n$. Then the number of permutation of n things by taking all the things at a time is given by $\frac{n!}{n_1!\,n_2!\dots n_r!}$

Problems:-

1. Find the number of permutations of the Letter of the following words.

(i) PROGRESS (ii) MATHEMATICS (iii) TOPOLOGY

(iv) Engineering (v) Vidhana soudha (vi) Anuragga

Soln: (i) PROGRESS word has 8 Letters ($n=8$)

It has R:2, S=2, P, O, G, E = 1

$n_1+n_2+n_3+n_4+n_5+n_6 = 8$

The required number of permutation is given by

$$= \frac{8!}{2!\,2!\,(1!)^4} = \frac{40320}{4} = 10080$$

(ii) MATHEMATICS word has 11 Letters ($n=11$)

Here M=2, A=2, T=2, H, E, S, I, C = 1

The req. no. of permutation is, $\frac{11!}{2!\,2!\,2!\,(1!)^5}$

$$= \frac{39916800}{8} = 4989600$$

(iii) TOPOLOGY ($n=8$)

Also O=3, TPLGY = 1

The req. no. of permutations $= \frac{8!}{3!\,(1!)^5}$

$$= \frac{40320}{6} = 6720$$

(iv) ENGINEERING word has 11 Letters $(n=11)$

Here $E=3$, $N=3$, $I=2$, $G=2$, $R=1$

The req. no. of permutation is, $\dfrac{11!}{3!\,3!\,2!\,2!\,1!}$

$$= \frac{39916800}{144} = 277200$$

V) VIDHANA SOUDHA

Here $n=13$

$V=1$, $I=1$, $D=2$, $H=2$, $A=3$, $N=1$, $S=1$, $O=1$, $U=1$

The req. no. of permutations is, $\dfrac{13!}{(1!)^6\,(2!)^2\,(3!)}$

$$= \frac{6227020800}{24} = 259459200$$

VI) ANURAAGA has $n=8$

$A=4$, $N=1$, $U=1$, $R=1$, $G=1$

The req. no. of permutations is $\dfrac{8!}{4!\,(1!)^4} = 1680$

2. Find the number of permutations of the letter of the word ENGINEERING such that

i) all the E's are together

ii) G and R are next to each other

iii) All the vowels are adjacent

iv) Arrangement begin with N.

Sol^n. ENGINEERING has $n=11$

Here $E=3$, $N=3$, $I=2$, $G=2$, $R=1$

i) All the E's are together

The 3 E's is to be treated as a single Letter

(EEE) NNN II GG R $(n=1+8=9)$

∴ The req. no. of permutation is, $\dfrac{9!}{1!\,3!\,2!\,2!\,1!}$

$$= \frac{362880}{24} = 15120$$

(ii) G and R are next to each other

Both together as a single letter then remaining

$E=3, N=3, I=2 \quad (n=1+8=9)$

The req. no. of permutations is $\frac{9!}{3!\,3!\,2!\,1!}$

$= \frac{362880}{72} = 5040$

But there are 2 G's and 1 R

we have GGR, GRG, RGG equal to 3.

$\therefore$ The req. no. of permutations is 5040×3

$= 15120$

(iii) All the vowels are adjacent

Letter A, E, I, O, U are vowels

In the word vowels treat E & I as a single letter. Then we have (EEEII) NNNGGR $(n=1+6=7)$

The associated no. of permutations $= \frac{7!}{1!\,3!\,2!\,1!}$

$= \frac{5040}{12} = 420$

But we observe 3 E's & 2 I's

They can be arranged in $\frac{5!}{3!\,2!} = 10$ ways

The req. no. of permutations $= 420 \times 10 = 4200$

(iv) Arrangements begin with N.

Since we have 11 Letters, we left with 10 blank spaces for filling

$E=3, \; I=2, \; G=2, \; R=1, \; N=(3-1)=2$

The req. no. of permutations $= \frac{10!}{3!\,2!\,2!\,1!\,2!}$

$= \frac{3628800}{48} = 75600$

3. Find the number of permutations of the letters of the word MISSISSIPPI. How many of these

(i) Begin with the letter I

(ii) Begin and end with S

(iii) has all I's together.

<u>Soln</u>: MISSISSIPPI has 11 letters $(n=11)$

Also $M=1,\ I=4,\ S=4,\ P=2$

The required no. of permutations $= \dfrac{11!}{1!\,4!\,4!\,2!}$

$= 34650$

(i) Begin with the Letter I

Since 11 Letters, we left with 10 blank spaces for filling we have $M=1,\ I=4-1=3,\ S=4,\ P=2$

$\therefore$ The req. no. of permutation $= \dfrac{10!}{1!\,3!\,4!\,2!} = 12600$

(ii) Begin and end with S

we are left with 9 blank spaces we have

$M=1,\ I=4,\ S=4-2=2\quad P=2$

$\therefore$ The req. no. of perm $= \dfrac{9!}{1!\,4!\,2!\,2!} \times 2$

$= 7560$

(since first & Last letter is to be S, so x by 2)

(iii) All the I's together

The 4 I's is to be treated as a single letter

(IIII) M SSSS PP $(n=1+7=8)$

The req. no. of perm $= \dfrac{8!}{1!\,1!\,4!\,2!} = 840$

✓ Imp

4. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

Sol^n: The no. of digits are 7

Let $n = d_1, d_2, d_3, d_4, d_5, d_6, d_7$ d_i's being digits

If n has to exceed 5,000,000 which also has 7 digits, it is necessary that digit $d_1 = 5$ or 6 or 7

Case(I): suppose $d_1 = 5$ the rest of the 6 digits has 2 Nos of 4's, 1 each of 3, 5, 6, 7

$\therefore$ The req. no. of perm $= \dfrac{6!}{2!\,(1!)^4} = 360$

Case (II): suppose $d_1 = 6$, the rest of 6 digits has 2 No's of 4, 2 No's of 5, 1 each of 3 & 7

The no. of such perm $= \dfrac{6!}{(2!)^2\,(1!)^2} = 180$

Case(III): suppose $d_1 = 7$, The rest of 6 digits has 2 No's of 4, 2 No's of 5, 1 each of

$\therefore$ The No. of perm. $\dfrac{6!}{(2!)^2\,(1!)^2} = 180.$

By applying sum rule the req. number $n = 360 + 180 + 180 = 720$ it exceeding 5,000,000 is the sum of all these.

5. How many numbers greater than 1,000,000 can be formed by using the digits 1, 2, 2, 2, 4, 4, 0?

Sol^n: The no. of digits are given 7 & 1,000,000 also has 7. In order to have the no. > 1000000 the no. has to begin with 1 or 2 or 4

The no. begin with 1 $= \dfrac{6!}{3!\,2!} = 60$

" 2 $= \dfrac{6!}{2!\,2!} = 180$

" 4 $= \dfrac{6!}{3!} = 120.$

By sum rule $60 + 180 + 120 = 360$

Thus req. no. $> 1,000,000$ is 360

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The Binomial Theorem

If 'n' is a +ve integer, we are already familiar with the expansion of $(x+y)^n = x^n + nC_1 x^{n-1} y + nC_2 x^{n-2} y^2 + nC_3 x^{n-3} y^3 + \dots + nC_r x^{n-r} y^r + \dots + y^n$ —— (1).

It may be this

There are $(n+1)$ terms in the expansion, the general term is T_{r+1} ⟵ (middle term → is in b/w the series)

where $T_{r+1} = nC_r x^{n-r} y^r = nC_{n-r} x^r y^{n-r}$ —— (2).

Eq^n (1) in summation form

$$(x+y)^n = \sum_{r=0}^{n} nC_r x^{n-r} y^r = \sum_{r=0}^{n} nC_r x^r y^{n-r} \quad \text{—— (3).}$$

Generalised Binomial theorem or Multinomial theorem:

If $n_1, n_2, \dots, n_r$ are +ve integers

$$(x_1 + x_2 + x_3 + \dots + x_r)^n = \sum \binom{n}{n_1\, n_2 \dots n_r} x_1^{n_1} x_2^{n_2} \dots x_r^{n_r} \quad \text{—— (4)}$$

where $\binom{n}{n_1\, n_2 \dots n_r} = \dfrac{n!}{n_1!\, n_2! \dots n_r!}$

Corollary:-

1. Let $x=1, y=1$ in (3)

$$(1+1)^n = \sum_{r=0}^{n} nC_r = nC_0 + nC_1 + nC_2 + \dots + nC_n$$

$$2^n = nC_0 + nC_1 + nC_2 + \dots + nC_n$$

∴ Sum of all the coefficients in the expansion of $(x+y)^n$ is 2^n

(2). Take $x_1 = 1, x_2 = 1, \dots x_r = 1$ in (4).

$$(1+1+\dots+r\text{ terms})^n = \sum \binom{n}{n_1\, n_2 \dots n_r}$$

$$r^n = \sum \binom{n}{n_1\, n_2 \dots n_r}$$

$\therefore$ Sum of all the coefficients in the expansion of $(x_1+x_2+x_3+\cdots+x_r)^n = r^n$.

~~Corollary~~:-

Examples:-

1. $\binom{5}{2\ 2\ 1} = \dfrac{5!}{2!\,2!\,1!} = \dfrac{120}{4} = 30$

2. $\binom{6}{1\ 2\ 3} = \dfrac{6!}{1!\,2!\,3!} = \dfrac{720}{12} = 60$

3. $\binom{7}{2\ 3\ 2} = \dfrac{7!}{2!\,3!\,2!} = \dfrac{5040}{24} = 210$

4. $\binom{8}{1\ 2\ 2\ 3} = \dfrac{8!}{1!\,2!\,2!\,3!} = \dfrac{40320}{24} = 1680$

5. $\binom{9}{3\ 3\ 3\ 0} = \dfrac{9!}{3!\,3!\,3!\,0!} = \dfrac{362880}{216} = 1680$

Problems:-

1. Find the following coefficients (i) $a^5 b^2$ in the expansion of $(2a-3b)^7$ (ii) x^6y^3 in the expansion of $(x+2y)^9$.

Soln:. we have $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$

$$(2a-3b)^7 = \sum_{r=0}^{7} {}^7C_r\,(2a)^{7-r}(-3b)^r$$

$$= \sum_{r=0}^{7} {}^7C_r\, 2^{7-r} a^{7-r} (-3)^r b^r$$

$$= {}^7C_2\, 2^5 a^5 (-3)^2 b^2$$

$$= {}^7C_2\, 2^5 (-3)^2 a^5 b^2$$

$\therefore$ The coeff. of $a^5 b^2 = 7C_2 \, 2^5 (-3)^2$

$$= \frac{7!}{2!\,5!} (32)(9)$$

$$= 6048$$

(ii) $(x+2y)^9 = \sum_{r=0}^{9} 9C_r \, x^{9-r} (2y)^r$

By taking $r=3$ we have

$$= \sum_{r=0}^{9} 9C_3 \, x^{9-3} (2y)^3$$

$$= 9C_3 \, x^6 \, 2^3 y^3$$

$\therefore$ The coeff of $x^6 y^3 = 9C_3 \, 2^3$

$$= \frac{9!}{3!\,6!} \, 8 = 672$$

2. Find the term independent of x and also the middle term in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^6$

Sol^n: we have $T_{r+1} = \cancel{\sum_{r=0}^{6} 6C_r \left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{6-r}}$

$$T_{r+1} = 6C_r \left(\frac{3x^2}{2}\right)^{6-r} \left(\frac{-1}{3x}\right)^r$$

$$= 6C_r \left(\frac{3}{2}\right)^{6-r} (x^2)^{6-r} \left(\frac{-1}{3}\right)^r \left(\frac{1}{x^r}\right)$$

$$= 6C_r \left(\frac{3}{2}\right)^{6-r} x^{12-2r} \left(\frac{-1}{3}\right)^r \cdot x^{-r}$$

$$= 6C_r \left(\frac{3}{2}\right)^{6-r} \left(\frac{-1}{3}\right)^r x^{12-3r}$$

To get the term independent of x put $r=4$.

i.e $12 - 3r = 0 \Rightarrow r = 4$.

$$6C_4 \left(\frac{3}{2}\right)^{6-4} \left(\frac{-1}{3}\right)^4 x^{12-12} = 6C_4 \left(\frac{3}{2}\right)^2 \left(\frac{-1}{3}\right)^4$$

$$= 6C_4 \frac{(3)^{\cancel{2}}}{4} \frac{1}{(3)^{\cancel{4}}}$$

$= 6C_4 \frac{1}{4} \cdot \frac{1}{3^2}$

$= \frac{5}{12}$ is the coeff of x independent term.

Further we note that $n=6$ there will be 7 terms in the expansion and 4^{th} term T_4 happens to be middle term. This term is obtained by taking $r=3$

$$T_4 = 6C_3 \left(\frac{3x^2}{2}\right)^3 \left(\frac{-1}{3x}\right)^3$$

$$= 20 \frac{27 x^6}{8} \frac{(-1)}{27x^3}$$

$$= \frac{-5}{2} x^3$$

Thus the required middle term is $-\frac{5}{2}x^3$

3. Find the coefficient of $x^2y^2z^3$ in the expansion of $(x+y+z)^7$. Also find the sum of coefficients

Sol: The general term is $\binom{n}{n_1\ n_2\ n_3} x^{n_1} y^{n_2} z^{n_3}$

where $(n_1\ n_2\ n_3) = (2\ 2\ 3)$ & $n=7$

$\therefore$ we have $\binom{7}{2\ 2\ 3} x^2y^2z^3$

The coeff of $x^2y^2z^3 = \frac{7!}{2!\,2!\,3!} = \frac{5040}{24} = 210$

Further the sum of all coeff in the exp. of $(x+y+z)^7$ is $(1+1+1)^7 = 3^7$

VImp

4. Determine the coeff of xyz^2 in the expansion of $(2x-y-z)^4$.

Sol: The general term is given by $\binom{4}{n_1\ n_2\ n_3} (2x^{n_1}) (-y)^{n_2} (-z)^{n_3}$

Taking $n_1=1,\ n_2=1,\ n_3=2.$

$\begin{pmatrix} 4 \\ 1\ 1\ 2 \end{pmatrix} (2x)^1 (-y)^1 (-z)^2 = \frac{4!}{1!\,1!\,2!} (-2xyz^2)$

$= -24\,xyz^2$

$\therefore$ The req. coeff of $xyz^2 = -24$.

5. Find the coeff of $a^3 b^2 c d^2$ in the expansion of $(2a-b+3c-2d)^8$

Sol^n: $\begin{pmatrix} 8 \\ n_1\ n_2\ n_3\ n_4 \end{pmatrix} (2a)^{n_1} (-b)^{n_2} (3c)^{n_3} (-2d)^{n_4}$

By taking $n_1 = 3$, $n_2 = 2$, $n_3 = 1$, $n_4 = 2$ we have

$\begin{pmatrix} 8 \\ 3\ 2\ 1\ 2 \end{pmatrix} (2a)^3 (-b)^2 (3c)^1 (-2d)^2$

$\Rightarrow \frac{8!}{3!\,2!\,1!\,2!}\, 2^3\, 3^1\, 2^2\, a^3 b^2 c d^2$

$\Rightarrow \frac{8!}{3!\,2!\,1!\,2!} \times 96$

$\Rightarrow 161280$.

The coeff of $a^3 b^2 c d^2$ is 161280

v.Imp 6. Find the coefficient of $x^{11} y^4$ in the expansion of $(2x^3 - 3xy^2 + z^2)^6$

Sol^n: The general term $= \begin{pmatrix} 6 \\ n_1\, n_2\, n_3 \end{pmatrix} (2x^3)^{n_1} (-3xy^2)^{n_2} (z^2)^{n_3}$

i.e $\begin{pmatrix} 6 \\ n_1\, n_2\, n_3 \end{pmatrix} 2^{n_1} (-3)^{n_2} x^{3n_1+n_2} y^{2n_2} z^{2n_3}$ —— ①

we have $3n_1 + n_2 = 11$, $2n_2 = 4 \Rightarrow n_2 = 2$ & $n_1 = 3$

Further $n_1 + n_2 + n_3 = 6$

$\Rightarrow n_3 = 1$

① $\Rightarrow \begin{pmatrix} 6 \\ n_1\, n_2\, n_3 \end{pmatrix} 2^3 (-3)^2 x^{11} y^4 z^2$

The coeff. of $x^{11}y^4 = \dfrac{6!}{3!\,2!\,1!} \times 8 \times 9z^2$

$= 4320\, z^2$

V Imp

7. Find the coeff. of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$

Sol^n: The general term is given by

$$\binom{16}{n_1\ n_2\ n_3\ n_4\ n_5} a^{n_1} (2b)^{n_2} (-3c)^{n_3} (2d)^{n_4} 5^{n_5}$$

We shall take $n_1 = 2$, $n_2 = 3$, $n_3 = 2$, $n_4 = 5$ & hence $n_5 = 4$

$$\binom{16}{2\ 3\ 2\ 5\ 4} a^2 (2b)^3 (-3c)^2 (2d)^5\, 5^4$$

$$\binom{16}{2\ 3\ 2\ 5\ 4} a^2 b^3 c^2 d^5\, 2^3 (-3)^2 2^5 5^4$$

$\therefore$ The coeff. of $a^2b^3c^2d^5$ is given by

$$\binom{16}{2\ 3\ 2\ 5\ 4} 2^3 (-3)^2 2^5 5^4$$

$$= \frac{16!}{2!\,3!\,2!\,5!\,4!}\ 256 \times 9 \times 625 = \frac{125}{6} \times 16!$$

$\therefore$ The req. coeff is $\frac{125}{6} \times 16!$

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Combination with repetitions:-

The number of combination of n distinct things taken r at a time with possible repetition is given by ${}^{n+r-1}C_r$

Using the property ${}^nC_r = {}^nC_{n-r}$

we have ${}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$

Things	combination without repetition	Combination with repetition.
a b c	Taken 2 at a time $(n=3,\ r=2)$ ab, bc, ca = 3 ${}^nC_2 = {}^3C_1 = 3$	Taken 2 at a time $n=3,\ r=2$ ab, bc, ca aa, bb cc $\}=6$ ${}^{n+r-1}C_r = {}^4C_2 = 6$
a b c d	Taken 3 at a time $(n=4,\ r=3)$ abc, abd, bcd, cda = 4 ${}^4C_3 = {}^4C_1 = 4$	Taken 3 at a time abc, abd, bcd, cda aad, aac, aad, bba bbc, bbd, cca, ccb, ccd, dda, ddb, ddc aaa, bbb, ccc, ddd $=20$ ${}^{n+r-1}C_r = {}^6C_3 = 20.$

Note:-

${}^{n+r-1}C_r$ represents the number of non-ve integer Sol^n of the following eq^n with n unknowns.

$x_1 + x_2 + x_3 + \dots + x_n = r.$

It also represents the no. of distinct terms in the exp. of $(x_1 + x_2 + \dots + x_n)^r$

Further ${}^{n+r-1}C_r$ also represents the various number of ways in the distribution of r identical things into n distinct boxes

Ex:-

1. A sweet stall has 12 types of sweets and there are atleast 10 sweets of each type. we shall find the no. of ways in which 10 sweets can be selected.

Here $n=12$ & $r=10$

The no. of ways of selection is $(n+r-1)C_r$ which being $21C_{10}$

2. Let us suppose that there are 8 pencils of same colour and size which should be put in 4 distinct pouches.

The no. of possible ways in $(n+r-1)C_r$ where $r=8$, $n=4$

The no. of possible ways $11C_8 = 11C_3 = 165$.

3. We shall find the no. of non-ve integer solⁿ of the eqⁿ $a+b+c+d=6$ OR

we shall find the no. of distinct terms in the expansion of $(a+b+c+d)^6$.

The required no. is where $n=4$, $r=6$

The no. of non -ve integer solⁿ $= 9C_6 = 9C_3 = 84$

This is equal to the distinct terms in the expⁿ of $(a+b+c+d)^6$.

Problems:-

1. In how many ways can 10 identical dimes be distributed among 5 children if

V.Imp

i) there are no restrictions

ii) Each child gets atleast one dime

iii) The atleast child gets atleast 2 dims.

Note: Dime is a 10 cent coin like 10 rupees coin.

Soln: (i) 10 identical dimes is to be distributed among 5 children. we have $r=10, n=5$

$$^{n+r-1}C_r = {}^{14}C_{10} = {}^{14}C_4 = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4 \cdot 3 \cdot 2 \cdot 1} = 1001$$

(ii) First we have to distribute 1 dime to each child with the result we are left with 5 dimes for 5 child

$r=5, n=5$

$$^{n+r-1}C_r = {}^{9}C_5 = {}^{9}C_4 = \frac{9 \cdot 8 \cdot 7 \cdot 6}{24} = 126$$

(iii) we have to give 2 dime to the oldest child with the result, left with 2 dimes for distribution among 5 children. Here $r=8, n=5$

$$^{n+r-1}C_r = {}^{12}C_8 = {}^{13}C_4 = 495$$ be the required no. of ways.

2. Find the number of ways in which 8 shirts and 4 sweaters are to be distributed among 5 selected old people in an old age home such that each one gets atleast one shirt.

Soln: Let us suppose that 1 shirt is given to all the 5 people. Here $r=3, n=5$

The no. of ways $^{n+r-1}C_r = {}^{7}C_3 = 35$

secondly 4 Sweaters to be distributed among 5

$r=4,\ n=5$ and the no. of ways

$$^{n+r-1}C_r = {}^{8}C_4 = 70$$

The no. of ways $= 35 \times 70 = 2450$.

v.Imp 3. In how many ways one can distributed 8 identical marbles in 4 distinct containers. Such that

i) No container is empty

ii) Fourth container has an odd number of marbles in it.

Sol^n. i) we have $n=4,\ r=4$

The no. of ways $^{n+r-1}C_r = {}^{7}C_4 = 35$ be the required no. of ways.

(ii) 4^{th} container can contain 1 or 3 or 5 or 7 marbles

This will give us 4 options. The computation of the no. of ways as required is

No. of marbles in 4^{th} container	Distributed among 3 container	No. of ways $^{n+r-1}C_r$
1	7 marbles among 3, $n=3,\ r=7$	$^{9}C_7 = {}^{9}C_2 = 36$
3	5 marbles among 3 container $n=3,\ r=5$	$^{7}C_5 = {}^{7}C_2 = 21$
5	3 marbles among 3 container $n=3,\ r=3$	$^{5}C_3 = {}^{5}C_2 = 10$
7	1 marble among 3 $n=3,\ r=1$	$^{3}C_1 = 3$

The req. no. of ways $= 36+21+10+3 = 70$.